Finite reflection groups, hyperplane arrangements, and (oriented) matroids

Michael Cuntz, Lukas Kühne, Raman Sanyal

(R. Sanyal) Institut für Mathematik, Goethe-Universität Frankfurt, Frankfurt am Main, Germany

Email address: sanyal@math.uni-frankfurt.de

(M. Cuntz) Leibniz Universität Hannover, Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Fakultät für Mathematik und Physik, Welfengarten 1, D-30167 Hannover, Germany

Email address: cuntz@math.uni-hannover.de

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CHAPTER 1

Finite reflection groups

1. The symmetric group

The **symmetric group** $\mathfrak{S}_n$ is set of bijections from $[n] := \{1, 2, \dots, n\}$ to itself. We can all agree that permutations and their combinatorics are ubiquitous throughout all of mathematics. Here we recall the basic properties of $\mathfrak{S}_n$ and view it as a group generated by reflections in hyperplanes.

We represent a permutation $\sigma : [n] \to [n]$ in **one-line** notation, which is the bottom row of the table

n	1	2	3	4	5	6	7	8	9
$\sigma(n)$	4	6	7	1	9	3	5	2	8

Thus $\sigma = 467193528$. The set of bijections becomes a group under composition. So, for example, for $\tau = 581639274$, the product with σ is $\sigma\tau = 924378651$. Alternatively, we write σ in **cycle notation** $\sigma = (14)(2637598)$. Fixed points, i.e., cycles of length 1 are usually omitted.

A **transposition** is a permutation that swaps two numbers i and j and leave all others fixed. We denote such a transposition by (i, j). Note that $(i, j)\sigma$ swaps the *numbers* i and j in the one-line notation, while $\sigma(i, j)$ swaps the entries at *positions* i and j.

$$\begin{aligned} (7,9)\sigma &= 469173528 \\ \sigma(7,9) &= 467193825 \end{aligned}$$

Also note that $\sigma(i, j) = (\sigma(i), \sigma(j))\sigma$ for all $i < j$.

The symmetric group is generated by **transpositions**. In fact, $\mathfrak{S}_n$ is generated by **adjacent** (or **simple**) transpositions, that is, the $n-1$ transpositions of the form $s_i = (i, i+1)$ for $i = 1, \dots, n-1$. For example,

$$\begin{aligned} \sigma &= s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_6 s_5 s_4 s_3 s_8 s_7 s_6 s_5 s_6 s_7 \\ &= (3,4)(2,3)(1,2)(5,6)(4,5)(3,4)(2,3)(6,7)(5,6)(4,5)(3,4)(8,9)(7,8)(6,7)(5,6)(6,7)(7,8) \end{aligned}$$

This is a **reduced expression**, in the sense that it uses the minimal number of simple transpositions. Reduced expressions are in general not unique. For σ there are 5630196 many distinct reduced expressions. The number of simple transpositions used is independent of the reduced expression and called the **length** $\ell(\sigma)$. An **inversion** of σ is a pair (i, j) with $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$. The **inversion number** $\operatorname{inv}(\sigma)$ is the number of inversions.

Exercise 1.1. Let $\sigma \in \mathfrak{S}_n$ be a permutation.

(1) Show that $\ell(\sigma) = \operatorname{inv}(\sigma)$.
(2) Find a procedure to generate all reduced expressions. ✎

We can view permutations as acting on $\mathbb{R}^n$. For $\sigma \in \mathfrak{S}_n$, define $P_\sigma \in \mathbb{R}^{n \times n}$ by

$$(P_\sigma)_{i,j} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases}$$

If $e_1, \dots, e_n$ are the standard basis vectors of $\mathbb{R}^n$, then $P_\sigma e_i = e_{\sigma(i)}$ and hence $P_\sigma P_\tau = P_{\sigma\tau}$. Note that $(P_\sigma)^{-1} = P_{\sigma^{-1}} = P_\sigma^t$ and hence P_σ is an *orthogonal* matrix with respect to the standard inner product

$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ on $\mathbb{R}^n$. So, for our σ we get

$$P_\sigma P_\tau \;=\; \begin{pmatrix} 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0 \end{pmatrix} \;=\; \begin{pmatrix} 0&0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{pmatrix} \;=\; P_{\sigma\tau}$$

Note that for a transposition (i,j), $P_{(i,j)}v = v$ if and only if $v_i = v_j$. That is, $P_{(i,j)}$ acts as the identity on the linear hyperplane $H_{ij} = \{v \in \mathbb{R}^n : v_i = v_j\}$. Moreover, $P_{(i,j)}$ is the reflection in the hyperplane H_{ij} in the sense that $v - P_{(i,j)}(v)$ is perpendicular to H_{ij}. Thus, we can view $\mathfrak{S}_n$ as a finite group of linear transformations that is generated by reflections in hyperplanes.

As a sneak peek, let us see how the geometry of the arrangement of the $\binom{n}{2}$ hyperplanes $H_{i,j}$ reflects properties of $\mathfrak{S}_n$. A point $v \in \mathbb{R}^n$ is not contained in any of the hyperplanes $H_{i,j}$ if and only if $v_i \neq v_j$ for all $i < j$. Hence, there is a unique permutation σ such that

$$v_{\sigma(1)} \;<\; v_{\sigma(2)} \;<\; \cdots \;<\; v_{\sigma(n)}\,.$$

Moreover, u is in the same connected component of $\mathbb{R}^n \setminus \bigcup_{i<j} H_{i,j}$ if and only if u is contained in the open convex set

$$C_\sigma \;=\; \{v \in \mathbb{R}^n : v_{\sigma(1)} < v_{\sigma(2)} < \cdots < v_{\sigma(n)}\}\,.$$

Thus

$$\mathbb{R}^n \setminus \bigcup_{i<j} H_{i,j} \;=\; \biguplus_{\sigma\in\mathfrak{S}_n} C_\sigma$$

and since $C_\sigma = P_\sigma(C_{\mathrm{id}})$, any two sets C_σ are isometric.

2. Reflections and reflection groups

Let V be an n dimensional real vector space with inner product $\langle .,.\rangle$. A **linear hyperplane** is a linear subspace of dimension $n-1$. That is, H is of the form

$$H = H_\alpha \;=\; \{v \in V : \langle \alpha, v\rangle = 0\}$$

for some $\alpha \in \mathbb{R}^n \setminus 0$. Note that α is unique up to scaling. The (orthogonal) **reflection** in H is the linear transformation $s_H : V \to V$ such that $s_H(v) = v$ for all $v \in H$ and $s_H(\alpha) = -\alpha$. This allows us to give an explicit description of s_H as

$$s_H(v) \;=\; v - \frac{2\langle\alpha, v\rangle}{\langle\alpha,\alpha\rangle}\alpha\,. \tag{1}$$

This does not depend on the choice of α but it is customary to write this as s_α. In particular $s_\alpha^2 = \mathrm{id}$ and s_α is an element of the orthogonal group $O(V) = \{g \in \mathrm{GL}(V) : \langle g(u), g(v)\rangle = \langle u, v\rangle$ for all $u, v \in V\}$.

A **finite reflection group** is a finite group of linear transformations $W \subseteq \mathrm{GL}(V)$ that is generated by (orthogonal) reflections in hyperplanes.

If W is generated by a single reflection, then $W = \{e, s_\alpha\}$.

To get a feeling, let us consider the case that W is generated by two reflections s_α and s_β. Now α, β are linearly independent and span a 2-dimensional subspace $L = \mathrm{span}\{\alpha, \beta\}$. Since $V = L \oplus L^\perp$, and s_α, s_β restrict to the identity on $L^\perp$, it suffices to consider the case $V = \mathbb{R}^2$. More generally, if W is a reflection group, then the **fixed space** is $V^W := \{u \in V : wu = u$ for all $w \in W\}$ and we can restrict the action of W to $(V^W)^\perp$. If $V^W = \{0\}$, then we call W **essential** (relative to V). The **rank** of W is the dimension $\dim(V^W)^\perp = \dim V - \dim V^W$. Hence, if W is minimally generated by 2 reflections, then W is of rank 2.

Let $\alpha, \beta \in \mathbb{R}^2 \setminus 0$. Let θ be the angle between H_α and H_β. The angle satisfies $\theta = \pi - \angle(\alpha,\beta) = \pi - \cos^{-1}\left(\frac{\langle\alpha,\beta\rangle}{\sqrt{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}}\right)$. Figure 1 shows the situation. If v has angle ω to the line H_α, then $\angle(v, s_\alpha(v)) = 2\omega$.

Likewise, if the angle between $s_\alpha(v)$ and H_β is γ, then $\angle(s_\alpha(v), s_\beta(s_\alpha(v))) = 2\gamma$. Since $\omega+\gamma=\theta$, we infer that $\angle(v, s_\beta(s_\alpha(v))) = 2\omega + 2\gamma = 2\theta$. Thus $s_\beta s_\alpha$ is a rotation by 2θ.

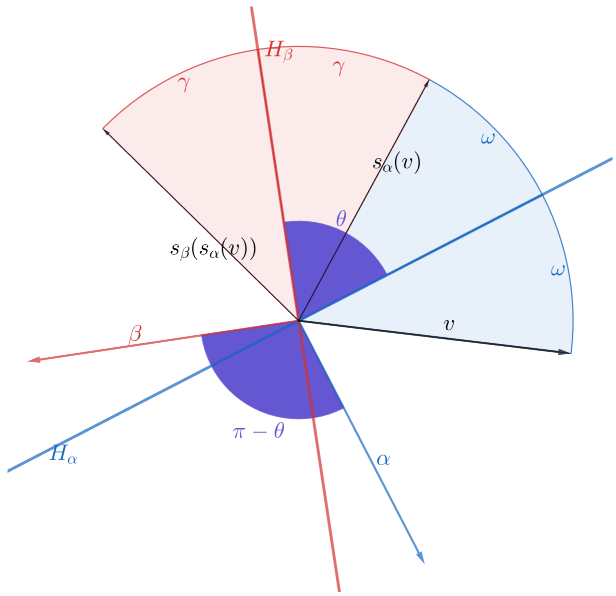


FIGURE 1. The composition of two reflections in the plane is a rotation.

Now if W is finite, then there is (a minimal) $m \geq 2$ such that $(s_\beta s_\alpha)^m = \mathrm{id}$. Such an m must satisfy $m\theta = \ell\pi$ for some $\ell \geq 1$.

Proposition 2.1. *Let H_α, H_β be hyperplanes with an angle $\frac{k}{\ell}\pi$ between them. Let W be the reflection group generated by s_α and s_β. Then there is a reflection $s_\gamma \in W$ whose hyperplane H_γ has an angle of $\frac{1}{\ell}\pi$ to H_α and s_α, s_γ generate W.*

PROOF. We can assume that k and ℓ are coprime and hence there is an $r > 0$ such that $rk = q\ell + 1$ for some q. Thus $w := (s_\beta s_\alpha)^r$ is a rotation by $\frac{2}{\ell}\pi$. Consider $w' := ws_\alpha \in W$. This is a reflection (check this!) in some hyperplane H_γ, i.e., $w' = s_\gamma$ and $w's_\alpha$ is a rotation of $\frac{2}{\ell}\pi$. Thus H_γ and H_α have an angle of $\frac{1}{\ell}\pi$ and $s_\beta s_\alpha = (s_\gamma s_\alpha)^k$, that is, $s_\beta = (s_\gamma s_\alpha)^k s_\alpha = (s_\gamma s_\alpha)^{k-1} s_\gamma$. ☐

We can always assume that W is generated by reflections in hyperplanes with angle $\frac{\pi}{m}$ between them.

If $m = 2$, then $\alpha \perp \beta$, that is, $\alpha \in H_\beta$ and $\beta \in H_\alpha$. Hence $s_\alpha s_\beta = s_\beta s_\alpha$ and $W = \{\mathrm{id}, s_\alpha, s_\beta\}$.

If $m \geq 3$, then pick $u_1 \in H_\alpha \setminus 0$. The orbit $Wu_1 = \{wu_1 : w \in W\}$ will consist of m points $u_1, u_2, \dots, u_m$ (why? ✎), which are the vertices of a regular polygon Q_m (why?? ✎). The group W is the symmetry group of Q_m, called the **dihedral group**, denoted by $\mathcal{D}_m$. The number of elements of $\mathcal{D}_m$ is $2m$.

Exercise 2.2. Answer the two questions marked ✎ in the paragraph above. ✎

This gives a complete classification of rank-2 reflection groups.

Notice that for any $m \geq 2$ if u is not contained in any of the reflection hyperplanes of W, then Wu are the vertices of a $2m$-gon that is not necessarily regular and for which W is not the symmetry group. We'll get back to these polygon much later.

Generally, if W is a finite reflection group and $s_\alpha, s_\beta \in W$ are two reflections, then the subgroup $W' = \langle s_\alpha, s_\beta \rangle$ generated by s_α, s_β is a finite reflection group of rank 2. Let us consider the various such subgroups for the symmetric group $\mathfrak{S}_n$. A normal for the hyperplane $H_{i,j}$ for $1 \leq i < j \leq n$ is given by $\alpha_{i,j} := e_j - e_i$. Thus, for two distinct transpositions $(i,j), (k,l)$, we have $\alpha_{i,j} \perp \alpha_{k,l}$ iff $\{i,j\} \cap \{k,l\} = \varnothing$. Otherwise

$|\{i,j\} \cap \{k,l\}| = 1$ and the angle between $H_{(i,j)}$ and $H_{(k,l)}$ is $\frac{\pi}{3}$. In particular, $\mathfrak{S}_3$ is the symmetry group of the regular triangle embedded in $\mathbb{R}^3$ with vertices $(1,0,0), (0,1,0)$, and $(0,0,1)$.

The symmetric group $\mathfrak{S}_n$ fixes the linear subspace $\{(t,t,\dots,t) : t \in \mathbb{R}\}$ pointwise and is of rank $n-1$. In light of the classification of finite reflection groups that we will be seeing in a few lectures, let us start by calling $\mathfrak{S}_n$ a reflection group of **type** A_{n-1}.

We close this first lecture with introducing one more example. For $i = 1,\dots,n$, let $z_i : \mathbb{R}^n \to \mathbb{R}^n$ be reflection in the hyperplane $H_i = \{v \in \mathbb{R}^n : v_i = 0\}$. That is, z_i flips the sign of the i-th coordinate of v. Consider the group W that is generated by $s_{(i,j)}$ for $1 \le i < j \le n$ as well as $z_1,\dots,z_n$. This is a group generated by reflections but it's not obvious that W is finite. To see that W is finite, consider the set $S = \{-1,+1\}^n$ of ± 1-vectors of length n. The subgroup $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ of linear symmetries of S is finite (why??) and W is clearly a subgroup. That makes W finite as well.

Exercise 2.3. Let $S \subset \mathbb{R}^n$ a finite set such that $\mathrm{span}(S) = \mathbb{R}^n$. Let G be the group of linear transformations $g \in \mathrm{GL}(\mathbb{R}^n)$ such that $gS = S$. Show that G is a finite group. ✎

Note that W contains more reflections then the ones we used to generate. Indeed, $z_i s_{(i,j)} z_i$ maps e_i to $-e_j$ and e_j to $-e_i$ and leaves e_k fixed for $k \notin \{i,j\}$. But this means it fixes all points of the hyperplane $H = \{v \in \mathbb{R}^n : v_i = -v_j\}$ and maps its normal $e_i + e_j$ to its negative. The finite reflection group W is of rank n and we say that it is of type $\mathbf{B_n}$.

Exercise 2.4. Show that the reflection group of type B_n is the symmetry group of the n-dimensional cube $C_n = \{v \in \mathbb{R}^n : |v_i| \le 1 \text{ for all } i = 1,\dots,n\}$. ✎

Exercise 2.5. A convex polytope P in $\mathbb{R}^3$ is the inclusion-minimal convex set containing a given finite set of points. The symmetry group G of P is the group of linear transformations $g \in \mathrm{GL}(\mathbb{R}^3)$ such that $gP = P$.

The boundary of P consists of vertices, edges, and faces. A **flag** of P is a choice of a vertex v, an edge e, and a face F such that $v \in e \subset F$. A convex polytope P is called **regular** if for two flags $v \in e \subset F$ and $v' \in e' \subset F'$ there is $g \in G$ such that $gv = v', ge = e'$, and $gF = F'$. Show that if P is regular, then G is a finite reflection group. ✎

Exercise 2.6. Consider two infinitely long walls meeting in a corner at an angle $\alpha \in (0,\pi]$. Show that any kicked ball (which doesn't loose momentum) can meet the walls only a finite number of times. What is the maximal number of times a ball can hit the walls? ✎

3. Root systems

Let W be a finite reflection group acting on V and let $\mathcal{A}(W)$ be the collection of reflecting hyperplanes of W. If $H_\alpha \in \mathcal{A}(W)$ and $w \in O(V)$, then $ws_\alpha w^{-1}$ sends $w\alpha$ to its negative and pointwise fixes $wH_\alpha = \{v \in V : \langle \alpha, w^{-1}v\rangle = \langle w\alpha, v\rangle = 0\} = H_{w\alpha}$. Here we used the fact that $w \in O(V)$ and hence $w^* = w^{-1}$. If $w \in W$, then $ws_\alpha w^{-1} \in W$, we infer that $H_{w\alpha} \in \mathcal{A}(W)$. Hence $w\mathcal{A}(W) = \mathcal{A}(W)$ for all $w \in W$. Instead of the action of W on the reflection hyperplanes, it is customary to consider the action on a collection of normal vectors to the reflection hyperplanes. Since $s_\alpha(\alpha) = -\alpha$ it is not sufficient to pick a normal vector for each hyperplane.

A **root system** Φ is a finite non-empty subset of $V \setminus 0$ such that for every $\alpha \in \Phi$

(R1) $\Phi \cap \mathrm{span}(\alpha) = \{-\alpha, \alpha\}$, and
(R2) $s_\alpha(\beta) \in \Phi$ for all $\beta \in \Phi$.

The elements of Φ are called **roots**. The **rank** of Φ is the dimension of $\mathrm{span}(\Phi)$.

Every finite reflection group W gives rise to a (actually many) root system:

$$\Phi \;=\; \{\alpha \in V : \langle \alpha, \alpha\rangle = 1, s_\alpha \in W\}$$

Conversely, if Φ is a root system, then define W as the subgroup of $O(V)$ generated by the reflections $\{s_\alpha : \alpha \in \Phi\}$. We may assume that $V = \operatorname{span}(\Phi)$ and appealing to Exercise 2.5 shows that W is a finite group.

Clearly if Φ is a root system, then $t\Phi = \{t\alpha : \alpha \in \Phi\}$ is a root system as well for all $t \in \mathbb{R} \setminus 0$. We call Φ **reducible** if there is a partition $\Phi = \Phi' \uplus \Phi''$ such that $\Phi' \perp \Phi''$ and Φ' and Φ'' are root systems. If no partition exists, then Φ is **irreducible**.

Exercise 3.1. Let Φ be a root system. For any non-empty $U \subseteq \Phi$, show that $\Phi \cap \operatorname{span}(U)$ is a root system. ✎

Exercise 3.2. Let Φ be an irreducible root system. Show that there are at most two different lengths of roots. (Hint: Show this first for root systems of rank 2.) ✎

Example 3.3 (A root system for type A_{n-1} and type B_n)**.** For type A_{n-1}: The set $\Phi = \{e_i - e_j : i, j \in [n], i \neq j\}$ is a root system for the symmetric group and since $\mathfrak{S}_n$ acts transitively on Φ, Φ is unique up to scaling.

For type B_n: The set $\Phi = \{\pm(e_i - e_j), \pm(e_i + e_j) : 1 \leq i < j \leq n\} \cup \{\pm e_1, \dots, \pm e_n\}$ is a root system whose associated reflection group is that of type B_n (check this!). Note that the roots have lengths 1 and $\sqrt{2}$. Since W acts by orthogonal transformations, there is more than one orbit of Φ under W. ◇

We call $c \in V$ **generic** relative to Φ if $\langle c, \alpha \rangle \neq 0$ for all $\alpha \in \Phi$. Since Φ is finite, generic c's exist.

Exercise 3.4. Identifying $V = \mathbb{R}^n$, show that there are infinitely many $t \in \mathbb{R}$ such that $c = (1, t, t^2, \dots, t^n)$ is generic for Φ. ✎

A **positive system** of Φ is a subset $\Phi^+ \subset \Phi$ of the form

$$\Phi^+ \;=\; \{\alpha \in \Phi : \langle c, \alpha \rangle > 0\}$$

for some generic c.

Clearly $|\Phi^+| = \frac{1}{2}|\Phi|$, as Φ^+ selects one element from each pair $-\alpha, \alpha \in \Phi$. But our choice serves a greater purpose. We seek to find smallest sets of reflections that generate W. As it will turn out, these subsets will be as small as possible.

Example 3.5 (Positive systems for type A_{n-1} and B_n)**.** For type A_{n-1}: A vector c is generic relative to $\Phi = \{e_i - e_j : i, j \in [n], i \neq j\}$ if and only if $c_i \neq c_j$ for all $i \neq j$. Consider $c = (c_1 < c_2 < \cdots < c_n)$. Then $\Phi^+ = \{e_j - e_i : 1 \leq i < j \leq n\}$.

For type B_n: c is generic if $|c_i| \neq |c_j|$ for all $i \neq j$ and $c_i \neq 0$ for all i. For $c = (0 < c_1 < \cdots < c_n)$, the positive system is $\Phi^+ = \{e_j - e_i, e_i + e_j : i < j\} \cup \{e_1, \dots, e_n\}$. ◇

For a finite set $U = \{u_1, \dots, u_m\} \subset V$, we define the **conical hull** as the set

$$\operatorname{cone}(U) \;=\; \{\mu_1 u_1 + \mu_2 u_2 + \cdots + \mu_m u_m : \mu_1, \dots, \mu_m \geq 0\}\,.$$

We call a finite set $U \subset V$ **acyclic** if there is $c \in V$ with $\langle c, u \rangle > 0$ for all $u \in U$.

Theorem 3.6. *Let $U \subset V$ be acyclic. If $U \cap \operatorname{span}(u) = \{u\}$ for all $u \in U$, then there is a unique inclusion-minimal $G \subseteq U$ with $\operatorname{cone}(G) = \operatorname{cone}(U)$.*

Proof. Let $G = \{g_1, \dots, g_k\} \subset U$ be an inclusion-minimal generating set for $C := \operatorname{cone}(U)$. That is, $C = \operatorname{cone}(G)$ but $C \neq \operatorname{cone}(G \setminus g_j)$ for all $j = 1, \dots, k$. Assume that $H = \{h_1, \dots, h_l\} \subset U$ also generates C but, say, $g_1 \notin H$. Then there are $a_1, \dots, a_l \geq 0$ such that $g_1 = \sum_i a_i h_i$ at least two a_i nonzero. On the other hand, there are $b_{ij} \geq 0$ such that $h_i = \sum_j b_{ij} g_j$. Setting $d_j = \sum_i a_i b_{ij} \geq 0$, we get

$$g_1 \;=\; d_1 g_1 + d_2 g_2 + \cdots + d_k g_k\,.$$

Said differently, $(1-d_1)g_1$ is contained in $\mathrm{cone}(G \setminus g_1)$. We cannot have $1-d_1 > 0$, as this would contradict the assumption that G is inclusion-minimal. If $1-d_1 < 0$, then

$$0 > (1-d_1)\langle c, g_1\rangle = \sum_{i=2}^{k} d_i \langle c, g_i\rangle \ \geq \ 0\,.$$

This leaves $d_1 = 1$. But the same calculation then shows $d_2 = \cdots = d_k = 0$, which implies $g_1 \in S$. Hence $G \subseteq U$. □

By Theorem 3.6, there is a unique inclusion-minimal subset $\Delta \subseteq \Phi^+$ that minimally generates $\mathrm{cone}(\Phi^+)$. We call Δ a **simple system**. Figure 2 shows two examples of root systems, positive systems, and simple systems in rank 2.

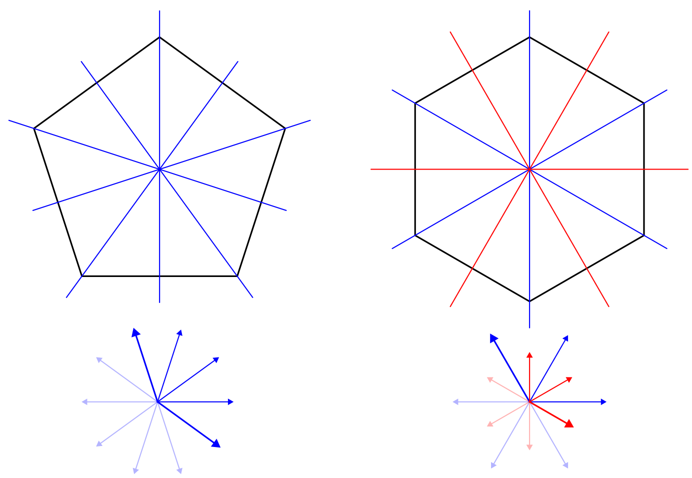


FIGURE 2. Two examples of finite reflection groups in the plane. Below root systems are shown. The positive system is opaque. The simple roots are slightly thicker.

We start with a seemingly technical but important lemma about simple systems.

Lemma 3.7. *Let Δ be a simple system. Then $\langle \alpha, \beta\rangle \leq 0$ for all $\alpha, \beta \in \Delta, \alpha \neq \beta$.*

PROOF. Let $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$. Then $L = \mathrm{span}(\alpha, \beta)$ is a 2-dimensional subspace and Exercise 3.1 yields that $\Phi \cap L$ is a root system with positive system $\Phi^+ \cap L$. Let $\Delta' \subseteq \Phi^+ \cap L$ be a simple system. If $\alpha \notin \Delta'$, then $\Delta' \cup (\Delta \setminus \alpha)$ generates $\mathrm{cone}(\Phi^+)$, contradicting Theorem 3.6. Hence $\alpha, \beta \in \Delta'$ and, using that L is 2-dimensional, $\Delta' = \{\alpha, \beta\}$. In particular, α, β are linearly independent.

Now assume that $\langle \alpha, \beta\rangle > 0$. We know that $\gamma = s_\alpha(\beta)$ is contained in $\Phi \cap L$. Now $s_\alpha(\beta) = \beta + \mu\alpha$, where $\mu = \frac{-2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} < 0$. But γ or $-\gamma$ is contained in $\Phi^+ \cap L$ but the unique expression of γ as a linear combination of α and β has positive and negative coefficients. This contradicts that $\Phi^+ \cap L \subseteq \mathrm{cone}(\alpha, \beta)$. □

A main consequence of this is the following.

Corollary 3.8. *Let Δ be a simple system, then Δ is linear independent. In particular, every root $\alpha \in \Phi$ has a unique expression in terms of simple roots, with all coefficients nonnegative or nonpositive.*

PROOF. Let $\alpha_1, \ldots, \alpha_m \in \Delta$ be a minimally dependent subset. Then there are $a_1, \ldots, a_m \in \mathbb{R}$ not all zero such that $0 = \sum_i a_i \alpha_i$. Since $\langle c, \alpha_i\rangle > 0$ for some generic c that brought us Φ^+, not all a_i are of the same sign. We may assume that $a_1, \ldots, a_k < 0 < a_{k+1}, \ldots, a_m$. Then

$$-\sum_{i=1}^{k} a_i \alpha_i \ = \ \sum_{j=k+1}^{m} a_j \alpha_j \ =: \ v$$

and minimality yields $v \neq 0$. We compute

$$0 \ < \ \langle v, v\rangle \ = \ \sum_{i=1}^{k}\sum_{j=k+1}^{m}(-\alpha_i)\alpha_j\langle\alpha_i,\alpha_j\rangle \ \leq \ 0\,,$$

which is a contradiction. □

The corollary furnishes another characterization of simple systems.

Corollary 3.9. *Let $S \subseteq \Phi$ be a basis for $\operatorname{span}(\Phi)$ such that $\Phi \subset \operatorname{cone}(S) \cup \operatorname{cone}(-S)$, then S is a simple system.*

Proof. We may assume that $\operatorname{span}(\Phi) = V$. Then $S = \{\alpha_1, \ldots, \alpha_n\}$ is a basis and there is some $c \in V$ with $\langle c, \alpha_i\rangle > 0$ for all i. As every $\beta \in \Phi$ is of the form $\beta = \sum_i a_i\alpha_i$ for some α_i that are all nonnegative or nonpositive, this shows that c is generic. In particular $\beta \in \Phi^+$ if and only if $\beta \in \operatorname{cone}(S) \cap \Phi$, which shows that S is a simple system. □

Example 3.10 (Simple systems for types A_{n-1} and B_n)**.** Continuing Example 3.5, let us choose $c = (0 < c_1 < \cdots < c_n)$, then the positive system for type A_{n-1} is $\Delta = \{e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}\}$. Indeed, for any $e_j - e_i \in \Phi^+$ with $1 \leq i < j \leq n$ we have

$$e_j - e_i \ = \ (e_j - e_{j-1}) + (e_{j-1} - e_{j-2}) + \cdots + (e_{i+1} - e_i)\,.$$

Likewise, for type B_n, we obtain $\Delta = \{e_1, e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}\}$. In addition to the positive roots $e_j - e_i$ for $i < j$, we get e_1 and $e_j = e_j - e_{j-1} + e_{j-1}$ by induction on j. The positive roots $e_j + e_i$ are then obvious. ◇

The fact that every root has a unique representation in terms of Δ with nonzero coefficients of like sign is the key in showing that Δ yields a generating set for W.

Lemma 3.11. *Let $\Delta \subseteq \Phi^+ \subseteq \Phi$. For $\alpha \in \Delta$ and $\beta \in \Phi^+$, we have $s_\alpha(\beta) \in \Phi^+$ if $\beta \neq \alpha$ and $s_\alpha(\beta) = -\alpha$ otherwise.*

Proof. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, $\alpha = \alpha_1$, and $\beta = \sum_i a_i\alpha_i$ with $a_1, \ldots, a_n \geq 0$. If $\alpha \neq \beta$, then there is some $a_i > 0$ for $i \geq 2$. Now

$$s_\alpha(\beta) = \beta - \mu\alpha_1 = (a_1 - \mu)\alpha_1 + \sum_{i\geq 2} a_i\alpha_i\,.$$

Since $s_\alpha(\beta) \in \Phi$ and the expression of $s_\alpha(\beta)$ is unique with nonzero coefficients of like sign, the fact that $\alpha_i > 0$, forces $a_1 - \mu$ to be positive as well. □

Theorem 3.12. *Let W be a finite reflection group with root system Φ. Let $\Delta \subset \Phi$ be a simple system, then $\{s_\alpha : \alpha \in \Delta\}$ generates W*

Proof. Let W' be the subgroup of W generated by the reflections $\{s_\alpha : \alpha \in \Delta\}$. It suffices to show that for every $\beta \in \Phi^+$, there is $w \in W'$ such that $w\beta \in \Delta$. Indeed if $\beta = w^{-1}\alpha$ for $w \in W'$ and $\alpha \in \Delta$, then $s_\beta = s_{w^{-1}\alpha} = w^{-1}s_\alpha w$ and hence $s_\beta \in W'$.

For any $\gamma \in \Phi^+$ let $\gamma = \sum_{\alpha\in\Delta} c_\alpha\alpha$ be the unique representation with all $c_\alpha \geq 0$ and define the **height** as $\operatorname{ht}(\beta) = \sum_{\alpha\in\Delta} c_\alpha$. Pick $\gamma \in W'\beta \cap \Phi^+$ with $\operatorname{ht}(\gamma)$ minimal. We claim that $\gamma \in \Delta$.

Since $\gamma \neq 0$, we have

$$0 \ < \ \langle\gamma,\gamma\rangle \ = \ \sum_{\alpha\in\Delta} c_\alpha\langle\gamma,\alpha\rangle$$

and thus there is some $\alpha \in \Delta$ with $c_\alpha > 0$ and $\langle\gamma, \alpha\rangle > 0$. But by (1), $s_\alpha(\gamma) = \gamma - \mu\alpha$, where $\mu > 0$. If $s_\alpha(\gamma)$ is in Φ^+, then $\operatorname{ht}(s_\alpha(\gamma)) < \operatorname{ht}(\gamma)$, which would contradict our choice of γ. Hence $s_\alpha(\gamma) \in -\Phi^+$ and by Lemma 3.11, this implies $\gamma = \alpha$. □

Example 3.13 (Generation by simple reflections in types A_{n-1} and B_n)**.** Continuing Example 3.10, note that $\mathrm{ht}(e_j - e_i) = j - i$. Applying, for example, s_α for $\alpha = e_j - e_{j-1}$ or $\alpha = e_{i+1} - e_i$ maps $e_j - e_i$ to $e_{j-1} - e_i$ or $e_j - e_{i+1}$, both of which have lower height. Repeating this, the process terminates at some $e_k - e_{k-1}$. Notice that the root of minimal height in $W'\beta$ is not unique.

Let us write $s_{j,i}$ for $s_{e_j - e_i}$ and $s_i = s_{i+1,i}$. Then $s_{i+1} \cdots s_{j-1}(e_j - e_i) = e_{i+1} - e_i$ and we obtain

$$s_{e_j - e_i} \;=\; s_{j-1} \cdots s_{i+1} s_i s_{i+1} \cdots s_{j-1}$$

For example, for $e_3 - e_1$, we get $s_2(e_3 - e_1) = e_2 - e_1$ and $s_{3,1} = s_2 s_1 s_2$. $\diamond$

Exercise 3.14. Let $\Phi = \{e_i - e_j : i \neq j\}$ be the root system of type A_{n-1}. Every subsets $U \subseteq \Phi$ determines a directed graph $D = (V, A)$ on nodes $V = \{1, \ldots, n\}$ and arcs $A = \{(i,j) : e_i - e_j \in U\}$.

1. Show that U generates $\mathfrak{S}_n$ if and only if D is connected.
2. Show that any minimal generating set U has the same cardinality.
3. How many minimal generating sets U are there?
4. How many simple systems are contained in Φ? ✎

Exercise 3.15. Exercise 3.14 suggests that U minimally generates $\mathfrak{S}_n$ if and only if U is a basis for $\{v \in V : v_1 + \cdots + v_n = 0\}$. Show that $\mathfrak{S}_n$ is special in that respect. ✎

4. Reflection arrangements and the length function

Let us interpret what we did geometrically. Let W be a finite reflection group with an arbitrary but fixed root system Φ. Recall that the reflection arrangement associated to W is the collection of hyperplanes $\mathcal{A}(W) = \{H_\alpha : \alpha \in \Phi\}$. A vector $c \in V$ is generic if and only if c is not contained in any of the hyperplanes in $\mathcal{A}(W)$. Let Φ_0^+ be an arbitrary but fixed positive system. For $c \in V \setminus 0$, we write $H_c^{\geq} := \{v : \langle c, v\rangle \geq 0\}$ for the **positive halfspace** that is bounded by the hyperplane H_c and $H_c^{>} := H_c^{\geq} \setminus H_c$ for the **open** halfspace. Now $c \in V \setminus 0$ yields Φ_0^+ if and only if $\Phi_0^+ = \Phi \cap H_c^{>}$, that is, $\langle \alpha, c\rangle > 0$ for all $\alpha \in \Phi_0^+$. More generally, for $v \notin \bigcup \mathcal{A}(W)$, let $\sigma_\alpha = \mathrm{sgn}\langle \alpha, v\rangle \in \{-1, +1\}$ for all $\alpha \in \Phi_0^+$. Then the connected component of $V \setminus \bigcup \mathcal{A}(W)$ containing v is

$$C_\sigma^\circ \;:=\; \{v \in V : \langle \sigma_\alpha \alpha, v\rangle > 0 \text{ for all } \alpha \in \Phi_0^+\}$$

and yields the positive system $\{\sigma_\alpha \alpha : \alpha \in \Phi_0^+\}$. Thus, the connected components of

$$V \setminus \bigcup \mathcal{A}(W)$$

are in bijection to the positive (and hence simple) systems of W. We write C_σ for the topological closure of C_σ°. Since C_σ° is non-empty, we get

$$C_\sigma \;=\; \{v \in V : \langle \sigma_\alpha \alpha, v\rangle \geq 0 \text{ for all } \alpha \in \Phi_0^+\}\,.$$

The closures of the various connected components are called the **regions** or **chambers** of $\mathcal{A}(W)$, which we collect in $\mathcal{R}(W)$.

A set $C \subseteq V$ is a **polyhedral cone** if there are $\beta_1, \ldots, \beta_m \in V$ with

$$C \;=\; \{v : \langle \beta_i, v\rangle \geq 0 \text{ for all } i = 1, \ldots, m\} \;=\; H_{\beta_1}^{\geq} \cap H_{\beta_2}^{\geq} \cap \cdots \cap H_{\beta_m}^{\geq}\,.$$

The choice of β_i is not unique. Indeed, for $\mu_1, \ldots, \mu_m \geq 0$ define $\beta = \mu_1\beta_1 + \cdots + \mu_m\beta_m$. Then $\langle \beta, v\rangle \geq 0$ for all $v \in C$ and $C = C \cap H_\beta^{\geq}$. We call the cone C **simplicial** if there is a choice of linearly independent $\beta_1, \ldots, \beta_m$. It is easy to see that C is then linearly isomorphic to $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^{n-m}$ if $n = \dim V$. We call an arrangement of hyperplanes **simplicial** if all regions are simplicial.

Corollary 4.1. *For any finite reflection group W, the arrangement $\mathcal{A}(W)$ is simplicial.*

Proof. Let Φ^+ be a positive system with associated chamber $C = \{v : \langle \beta, v\rangle \geq 0 \text{ for all } \beta \in \Phi^+\}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi^+$ be the simple system. We claim that

$$C \;=\; \{v : \langle \alpha, v\rangle \geq 0 \text{ for all } \alpha \in \Delta\}\,,$$

which then yields the result using Corollary 3.8. Let C' be the right-hand side. Since $\Delta \subseteq \Phi^+$, we get $C \subseteq C'$. Now for any $\beta \in \Phi^+$, there are $b_1, \dots, b_n \geq 0$ such that $\beta = b_1\alpha_1 + \cdots + b_n\alpha_n$. If $c \in C'$, then $\langle \beta, c\rangle = b_1\langle \alpha_1, c\rangle + \cdots + b_n\langle \alpha_n, c\rangle \geq 0$, which shows $c \in C$. □

Example 4.2 (Regions of types A_{n-1} and B_n)**.** We already did most of the leg work at the end of Section 1. The chambers of type A_{n-1} are the cones

$$\{v \in \mathbb{R}^n : v_{\tau(1)} < v_{\tau(2)} < \cdots < v_{\tau(n)}\}$$

as τ varies over all permutations $\tau \in \mathfrak{S}_n$. Each cone is linearly isomorphic to $\mathbb{R}^{n-1}_{\geq 0} \times \mathbb{R}$. In type B_n, the simple systems in Example 3.10 shows that the regions are of the form

$$\{v \in \mathbb{R}^n : 0 < \rho_1 v_{\tau(1)} < \rho_2 v_{\tau(2)} < \cdots < \rho_n v_{\tau(n)}\}$$

as τ varies over all permutations $\tau \in \mathfrak{S}_n$ and $\rho_1, \dots, \rho_n \in \{-1, +1\}^n$. ◇

The group W acts on the set of regions $\mathcal{R}(W)$. For every $w \in W$, $w\Phi_0^+$ is also a positive system as $w\Phi_0^+ = \Phi \cap H^>_{wc}$. Likewise, if $\Delta_0 \subseteq \Phi_0^+$ is the associated simple system, then $w\Delta_0$ is a simple system for all $w \in W$. The next thing that we want to verify is that every positive (and hence simple) system is of the form $w\Phi_0^+$ for some $w \in W$. Stronger even, we want to show that W acts **simply transitive** on the chambers of $\mathcal{A}(W)$, that is:

THEOREM 4.3. *Let Φ^+ be a positive system. Then there is a unique $w \in W$ with $\Phi^+ = w\Phi_0^+$.*

Let $C \in \mathcal{R}(W)$ be a chamber. A hyperplane $H \in \mathcal{A}(W)$ is a **wall** of C if $\operatorname{span}(C \cap H) = H$. If $C = H^{\geq}_{\beta_1} \cap \cdots \cap H^{\geq}_{\beta_m}$ is simplicial and $\beta_1, \dots, \beta_m$ linearly independent, then H_{β_i} is a wall of C for each $i = 1, \dots, m$.

If H is a wall, then $F = C \cap H$ is a **facet** (or **panel**) of C. For every facet F of C there is a unique chamber $C' \in \mathcal{R}(W)$ with $F = C \cap C'$. We call C and C' **adjacent**. A hyperplane $H \in \mathcal{A}(W)$ **separates** a chamber C from C' if C and C' lie on different sides of H. If C, C' are adjacent, then $\operatorname{span}(C \cap C')$ is the unique hyperplane that separates C from C'. We call $C_1, C_2, \dots, C_k$ of distinct chambers a **gallery** if C_i, C_{i+1} are adjacent for $1 \leq i < k$.

Proposition 4.4. *Let C, C' be regions of the reflection arrangement $\mathcal{A}(W)$. If C, C' are adjacent with separating hyperplane $H_\alpha \in \mathcal{A}(W)$. Then $s_\alpha(C) = C'$.*

PROOF. W acts on $\mathcal{A}(W)$ and hence on the regions $\mathcal{R}(W)$. In particular s_α fixes the facet $C \cap C' \subset H_\alpha$ pointwise. As s_α exchanges $H^<_\alpha$ and $H^>_\alpha$, this proves the claim. □

PROOF OF THEOREM 4.3 – EXISTENCE. Let C_0 and C be the chambers corresponding to the positive systems Φ_0^+ and Φ^+, respectively. Pick points $p \in C_0^\circ, q \in C^\circ$ such that the segment $[p, q] = \{p + \lambda(q - p) : 0 \leq \lambda \leq 1\}$ does not meet any of the finitely many codimension-2 subspace $H \cap H'$ for $H, H' \in \mathcal{A}(W)$ and $H \neq H'$. Let $H_{\beta_1}, H_{\beta_2}, \dots, H_{\beta_k} \in \mathcal{A}(W)$ be the ordered sequence of hyperplanes that meet the segment $[p, q]$ in distinct points. This means that there are chambers $C_0, C_1, \dots, C_k = C$ that meet $[p, q]$ such that C_iC_{i-1} are adjacent and separated by H_{β_i} for $i = 1, \dots, k$. It follows from Proposition 4.4 that $C_i = s_{\beta_i}(C_{i-1})$ for $i = 1, \dots, k$ and hence $C = wC_0$ for $w = s_{\beta_k} \cdots s_{\beta_1}$; see Figure 3. On the level of positive systems, this means $\Phi^+ = w\Phi_0^+$. □

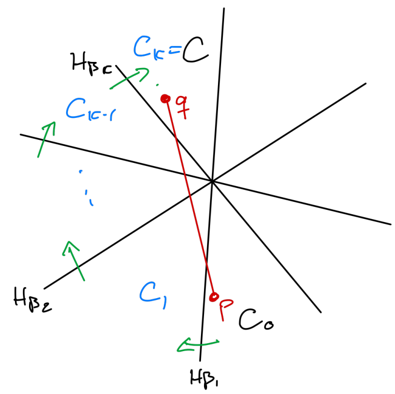


FIGURE 3. Illustration for proof of Theorem 4.3.

This proof paints a very geometric picture that we can also use to give an alternative proof of Theorem 3.12.

GEOMETRIC PROOF OF THEOREM 3.12. It suffices to show that every s_β for $\beta \in \Phi^+$ can be written as a product of simple reflections. As before let C_0 be the chamber corresponding to Δ_0 and pick points $p \in C_0^\circ$ and $q \in H_\beta$ such that the segment misses the finitely many codimension-2 intersections of hyperplanes in $\mathcal{A}(W)$. We again obtain a sequence of chambers $C_0, C_1, \ldots, C_k$ such that H_β is a wall for C_k (but for none of the other chambers). If $k = 0$, then H_β is a wall of C_0 and hence $\beta \in \Delta_0$ and s_β is a simple reflection. If $k > 0$, then let H_{β_i} be the hyperplane separating C_{i-1} from C_i and set $\beta_{k+1} = \beta$. Moreover, let $t_i = s_{\beta_i}$ be the reflection in H_{β_i}. Note that t_1 is a simple reflection, that $C_{i-1} = t_i(C_i)$ and hence $t_1 \cdots t_{i-1}(H_{\beta_i})$ is a wall of C_0. It follows that $s_i = t_1 t_2 \cdots t_{i-1} t_i t_{i-1} \cdots t_2 t_1$ is a simple reflection. By induction, we get that $t_i = t_{i-1} \cdots t_2 t_1 s_i t_1 t_2 \cdots t_{i-1}$ is a product of simple reflections. □

Let us harvest this geometric perspective further!

Construction 4.5. *If $w = s_1 s_2 \cdots s_r$ for simple reflections $s_i = s_{\alpha_i}$, then define*

$$\begin{aligned} t_1 &:= s_1 \\ t_2 &:= s_1\, s_2\, s_1 \\ &\vdots \\ t_r &:= s_1 s_2 \cdots s_{r-1}\, s_r\, s_{r-1} \cdots s_2 s_1 \,. \end{aligned}$$

Then each t_i is a reflection in the hyperplane H_{β_i} with $\beta_i = s_1 \cdots s_{i-1}\alpha_i$. Moreover

$$t_k t_{k-1} \cdots t_1 \;=\; s_1 s_2 \cdots s_k$$

for all $k = 1, \ldots, r$. Most importantly, if we define

$$C_k \;:=\; t_k(C_{k-1}) \;=\; t_k t_{k-1} \cdots t_1 C_0 \;=\; s_1 s_2 \cdots s_k C_0$$

for $k = 1, \ldots, r$, then $C_0, \ldots, C_r = wC$ is a gallery and t_i is the reflection in the unique hyperplane separating C_{i-1} and C_i.

As before let C_0 be the chamber corresponding to Φ_0^+. Note that $C_0 \subseteq H_\beta^{\geq}$ for all $\beta \in \Phi_0^+$. For every chamber C, let us write $\Phi_0^+(C) = \{\beta \in \Phi^+ : C \subseteq H_\beta^{\leq}\}$. That is, $\Phi_0^+(C)$ records the hyperplanes separating C from C_0. Moreover,

$$C \;=\; \bigcap_{\alpha \in \Phi_0^+(C)} H_\alpha^{\leq} \;\cap\; \bigcap_{\alpha \in \Phi_0^+ \setminus \Phi_0^+(C)} H_\alpha^{\geq}$$

If C corresponds to the positive system Φ^+, then $\Phi_0^+(C) = \Phi_0^+ \cap -\Phi^+$. We write $n(C) := |\Phi_0^+(C)|$ as the number of hyperplanes separating C from C_0.

Exercise 4.6. Let $\mathcal{A}$ be a collection of finitely many hyperplanes in V and $\mathcal{R} = \mathcal{R}(\mathcal{A})$ its set of regions. For any two regions $C, C' \in \mathcal{R}$ write $n(C.C')$ for the number of hyperplanes separating C from C'.

(1) Show that n defines a metric on $\mathcal{R}$.
(2) Let $p \in C^\circ$ and $q \in (C')^\circ$ be generic. Show that $n(C, C')$ is the number of hyperplanes $H \in \mathcal{A}$ that meet $[p, q]$.
(3) Let $C = C_0, C_1, \ldots, C_k = C'$ be a gallery and $H_i = \operatorname{span}(C_{i-1} \cap C_i)$ the walls along the gallery. Show that if $k > n(C, C')$, then there are $i < j$ such that $H_i = H_j$. ✎

Let C_0 be the chamber for Φ_0^+ and $w \in W$. We define $\Phi_0^+(w) := \Phi_0^+(wC_0)$ and $n(w) := n(wC_0)$.

Lemma 4.7. *Let $w = s_1 \cdots s_r$ such that $n(w) < r$, then there are $i < j$ such that*

$$w \;=\; s_1 \cdots \widehat{s}_i \cdots \widehat{s}_j \cdots s_r\,,$$

where $\widehat{s}_i$ and $\widehat{s}_j$ stands for omission.

Proof. Consider the gallery $C_0, \ldots, C_r$ of Construction 4.5 for $w = s_1 \cdots s_r$. If $r > n(w)$, then the gallery traverses a hyperplane twice by Exercise 4.6. That is, $t_i = t_j$ for some $i < j$ but this means

$$s_1 \cdots s_{i-1}\, s_i\, s_{i-1} \cdots s_1 \;=\; s_1 \cdots s_{i-1} s_i s_{i+1} \cdots s_{j-1}\, s_j\, s_{j-1} \cdots s_{i+1} s_i s_{i-1} \cdots s_1\,,$$

which reduces to

$$s_1 \cdots s_{i-1} \;=\; s_1 \cdots s_{i-1} s_i s_{i+1} \cdots s_{j-1}\, s_j\, s_{j-1} \cdots s_{i+1}\,,$$

which reduces to

$$s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} \;=\; s_1 \cdots s_{i-1} s_i s_{i+1} \cdots s_{j-1} s_j\,.$$

□

By Theorem 3.12 for every $w \in W$ there are $\alpha_1, \ldots, \alpha_k \in \Delta_0$ such that $w = s_{\alpha_1} \cdots s_{\alpha_k}$. If k is minimal, then we call $s_{\alpha_1} \cdots s_{\alpha_k}$ a **reduced expression** for w and define the **length** of w (relative to Δ_0) as $\ell(w) := k$.

Proposition 4.8. *Let $w \in W$.*

(i) *$\ell(e) = 0$ and $\ell(w) = 1$ if and only if $w = s_\alpha$ for some $\alpha \in \Delta_0$.*
(ii) *$\ell(w) = \ell(w^{-1})$.*
(iii) *$\ell(ww') \leq \ell(w) + \ell(w')$.*
(iv) *$\ell(s_\alpha w) = \ell(w s_\alpha) = \ell(w) \pm 1$ for all $\alpha \in \Delta_0$.*

Proof. (i) is clear. For (ii): If $w = s_{\alpha_1} \cdots s_{\alpha_k}$, then $w^{-1} = s_{\alpha_k} \cdots s_{\alpha_1}$, which shows $\ell(w^{-1}) \leq \ell(w)$. Since $w = (w^{-1})^{-1}$, we also get $\ell(w) \leq \ell(w^{-1})$. For (iii) just compose reduced expressions for w and w'.

For (iv), first note that $|\ell(s_\alpha w) - \ell(w)| \leq 1$. Now $\det(s_a) = -1$ (as H_α is the eigenspace to the eigenvalue 1 and $(H_\alpha)^\perp = \mathbb{R}\alpha$ is the eigenspace to eigenvalue -1). Thus $\det(w) = (-1)^{\ell(w)}$. This implies $\ell(s_\alpha w) \neq \ell(w)$. □

The next theorem gives a geometric interpretation of the length function.

Theorem 4.9. *Let W be a finite reflection group with simple positive systems $\Delta_0 \subseteq \Phi_0^+$. For all $w \in W$, we have $\ell(w) = n(w) = |\Phi_0^+ \cap w^{-1}(-\Phi_0^+)|$.*

Proof. Let $w = s_1 s_2 \cdots s_r$ be a reduced expression with simple reflections $s_1, \ldots, s_r$. Then there are at most r distinct hyperplanes in the gallery of Construction 4.5 that separate C_0 from $C_r = wC_0$. By Exercise 4.6, we obtain $n(w) = n(wC_0) \leq r = \ell(w)$. However, if $n(wC_0) < r$, then Lemma 4.7 yields $w = s_1 \cdots \widehat{s}_i \cdots \widehat{s}_j \cdots s_r$ which contradicts the assumption that the expression was reduced. □

Proof of Theorem 4.3 – Uniqueness. Assume that there is $w \in W$ such that $w\Phi_0^+ = \Phi_0^+$. But this means that $n(w) = 0$ and by Theorem 4.9 we get $w = e$. □

The following deletion condition follows directly from Theorem 4.9 and Lemma 4.7.

Corollary 4.10. *Let $w \in W$ and s_α a simple reflection.*

(1) $\ell(ws_\alpha) = \ell(w) + 1$ *if and only if* $w\alpha \in \Phi_0^+$.
(2) $\ell(ws_\alpha) = \ell(w) - 1$ *if and only if* $w\alpha \in -\Phi_0^+$.
(3) (Deletion Condition) *Let $w = s_1 \cdots s_r$ be a product of simple reflections. If the expression is not reduced, then there exist indices $1 \le i < j \le r$ with $w = s_1 \cdots \widehat{s}_i \cdots \widehat{s}_j \cdots s_r$.*
(4) (Exchange Condition) *Let $w = s_1 \cdots s_r$ be a product of simple reflections and s some simple reflection. If $\ell(ws) < \ell(w)$, then there is $1 \le i \le r$ with $w = s_1 \cdots \widehat{s}_i \cdots s_r s$. In particular, w has a reduced expression ending in s if and only if $\ell(ws) < \ell(w)$.*

Exercise 4.11. Proof Corollary 4.10. ✎

Exercise 4.12. Let W be a finite reflection group with simple system Δ and length function ℓ relative to Δ.

(1) Show that there is a unique element $w_0 \in W$ of maximal length. This is called the **longest element** of W relative to Δ. What is the length?
(2) Show that w_0 is an involution, that is $w_0^2 = e$.
(3) Prove that in every reduced expression of w_0, every simple reflection must occur at least once.
(4) Let w with reduced expression $w = s_1 \cdots s_r$. Show that there is a w' with reduced expression $w' = s_{r+1} \cdots s_m$ such that $s_1 \cdots s_r s_{r+1} \cdots s_m$ is a reduced expression for w_0.
(5) Show that for every $w \in W$ there is a simple system Δ such that w is the longest element.

✎

5. Fundamental domains and parabolic subgroups

For a group G acting on a space V, a **fundamental domain** is a *nice* (e.g. connected) subset $D \subset V$ such that D meets every orbit Gv for $v \in V$ in a unique point. Fundamental domains for arbitrary group actions might be tricky. For example, if G acts on $V = \mathbb{R}^2$ by rotation of $\frac{\pi}{2}$, then a fundamental domain is given by

$$D = \{(0,0)\} \cup \{(x,0) : x > 0\} \cup \{(x,y) : x, y > 0\}$$

see Figure 4. This set is neither open nor closed and it can be shown(?) that there is no nice connected fundamental domain for this action. The situation for finite reflection groups is rather different. If, for example, $W = \{e, s_\alpha\}$, then $H_\alpha^{\geq}$ or $H_\alpha^{\leq}$ acts as a fundamental domain, which is as nice as possible (closed, convex).

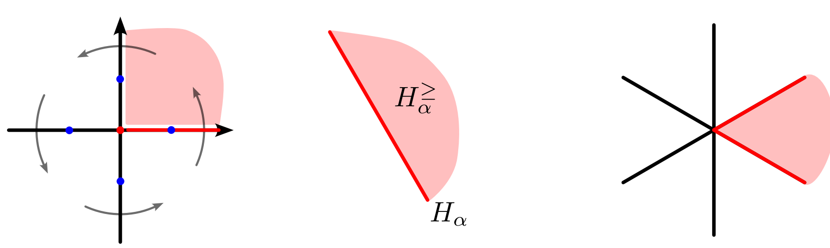
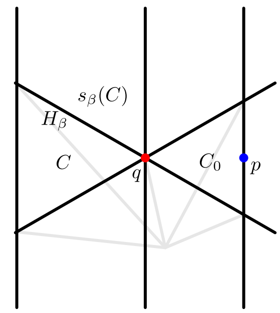


FIGURE 4. Fundamental domain for a rotation group acting on the plane. The blue points are in one orbit. The reddish point, half line and positive orthant are a fundamental domain.

THEOREM 5.1. *Let W be a finite reflection group. Then any $C \in \mathcal{R}(W)$ is a fundamental domain.*

PROOF. Let $C_0 \in \mathcal{R}(W)$ be a chamber with simple system Δ_0. For $v \in V$ we need to show that Wv meets C_0 in exactly one point. To see that there is $w \in W$ with $wv \in C_0$, let $C \in \mathcal{R}(W)$ with $v \in C$. By Theorem 4.3, there is a unique $w \in W$ with $C = wC_0$, that is, $C_0 = w^{-1}C$ and $w^{-1}v =: p \in C_0$.

Assume that there is $w' \in W$ with $w'v \in C_0$. This means that $p := w'v \in C_0$ and $wp =: q \in C_0$. If p is in the interior of C_0, then so is q and $wC_0 = C_0$ but this can only happen for $w = e$. Thus, $p, q \in \partial C_0$. We wish to prove that $p = q$.



Let $C = wC_0$. We prove the claim by induction on $\ell(w)$, the number of hyperplanes separating C_0 from C. Observe that $C_0 \cap C \neq \varnothing$ as both contain q. We claim that there is a wall H_β of C that separates C_0 from C and $q \in C \cap H_\beta$. If not, then every wall H of C that contains q satisfies $C_0 \subseteq H^{\geq}$ but this is absurd!. Now $C' = s_\beta(C)$ also contains q and hence $s_\beta wp = q$ and $\ell(s_\beta w) < \ell(w)$. The result now follows by induction. ☐

As an upshot of the proof, we can determine the **isotropy group** (or **stabilizer group**) $W_v := \{w \in W : wv = v\}$ for every point in V. For $v \in C_0$, the induction actually proves that if we let $J := \{\alpha \in \Delta_0 : v \in H_{\alpha_0}\}$, then W_v is generated by the reflections $\{s_\alpha : \alpha \in J\}$. Conversely for any $J \subseteq \Delta$, the subgroup $W_J = \langle s_\alpha : \alpha \in J\rangle$ is the isotropy group for all points in the linear subspace $L_J := \bigcap_{\alpha \in J} H_\alpha$. We call W_J a **standard parabolic subgroup** relative to the simple system Δ_0. For any $w \in W$, we call wW_Jw^{-1} a **parabolic subgroup**. That is, W' is a parabolic subgroup if W' is a standard parabolic subgroup for *some* simple system Δ.

Since for any $v \in V$, there is $w \in W$ with $wv \in C_0$, we deduce.

Corollary 5.2. *Every isotropy group is a parabolic subgroup.*

Exercise 5.3. Find an example of a reflection group W and a subgroup $W' \subset W$ that is generated by reflections but that is not a parabolic subgroup. ✎

We can make use of parabolic subgroups to study the **length generating polynomial**

$$W(q) \;=\; \sum_{w \in W} q^{\ell(w)}\,.$$

Example 5.4 (Dihedral groups)**.** For the dihedral group $\mathcal{D}_m$, we can compute by inspection

$$\mathcal{D}_m(q) \;=\; 1 + 2q + 2q^2 + \cdots + 2q^{m-1} + q^m$$

Let us define $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$, the **$q$-integer**. Then

$$\mathcal{D}_m(q) \;=\; (1+q)(1+q+q^2+\cdots+q^{m-1}) \;=\; [2]_q[m]_q\,.$$ ◇

To be able to compute $W(q)$, we want to use that for W_I, we can decompose W into cosets wW_I. If we can find suitable coset representative on which the length function is additive, then this might pave the way for an inductive procedure.

We defined W_I as the reflection group defined by a subset of simple reflections. It follows from Exercise 3.1 that $\Phi_I = \Phi \cap \operatorname{span}(I)$ is a root system for W_I with simple system I. Relative to I, we can define a length function ℓ_I on W_I.

Proposition 5.5. *We have $\ell_I(w) = \ell(w)$ for any $w \in W_I$.*

Proof. We recall that $\ell(w) = |\{\beta \in \Phi^+ : w\beta \in -\Phi^+\}|$. For $\beta \in \Phi^+ \setminus \Phi_I^+$ and $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$ with $c_\alpha \geq 0$, there is some $\gamma \in \Delta \setminus I$ with $c_\gamma > 0$. Thus, for all $\alpha \in I$, the coefficient of γ of $s_\alpha(\beta)$ is still positive. Thus, for $w \in W_I$ and $\beta \in \Phi^+$, this means $w\beta \in -\Phi^+$ if and only if $\beta \in \Phi_I^+$. Hence $\ell_I(w) = \ell(w)$. □

For $I \subseteq \Delta$ define

$$W^I \ := \ \{w \in W : \ell(ws_\alpha) > \ell(w) \text{ for all } \alpha \in I\}\,.$$

Lemma 5.6. *For every $w \in W$ there is a unique $u \in W^I$ and $v \in W_I$ such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$.*

Proof. For $w \in W$, let $u \in wW_I$ with $\ell(u)$ minimal and write $uv = w$ for $v \in W_I$. Let $u = s_1 \cdots s_k$ and $v = s_{k+1} \cdots s_{k+l}$ be reduced expressions. If $uv = s_1 \cdots s_k s_{k+1} \cdots s_{k+l}$ is not reduced, then by the Deletion Condition of Corollary 4.10 there are $1 \leq i < j \leq k + l$ such that s_i and s_j can be removed without altering uv. Now $j \leq k$ or $k < i$ would contradict the reducedness of the expressions for u and v. Thus $i < k < j$, but then this means that $u' = s_1 \cdots \widehat{s}_i \cdots s_k \in wW_I$ is of shorter length. Hence $\ell(w) = \ell(uv) = \ell(u) + \ell(v)$. If there is another $u' \in wW_I = uW_I$ with $\ell(u') = \ell(u)$, then $u = u'v'$ for some $v' \in W_I$ and the same argument shows $\ell(u) = \ell(u') + \ell(v')$. But then $\ell(v') = 0$ and $v' = e$.

Since u is chosen of minimal length in wW_I, this implies $\ell(us) > \ell(u)$ for all $s \in I$ and hence $u \in W^I$. □

The elements in W^I are called **minimal coset representatives**. For $X \subseteq W$ we define $X(q) := \sum_{w\in X} q^{\ell(w)}$. Then Lemma 5.6 implies

Corollary 5.7. *For any $J \subseteq \Delta$*

$$W(q) \ = \ W^J(q) W_J(q)\,.$$

Let us interpret W^I geometrically. Recall that $\ell(w^{-1}) = \ell(w)$ and $(ws_\alpha)^{-1} = s_\alpha w^{-1}$. Hence, we can write

$$W^I \ := \ \{w^{-1} : \ell(s_\alpha w) > \ell(w)\}\,.$$

The benefit of this small change is that according to Construction 4.5, $\ell(s_\alpha w) > \ell(w)$ if and only if wC_0 is *not* separated from C_0 by the hyperplane H_α. Hence W^I is in bijection to all chambers $C \in \mathcal{R}(W)$ with $C \subseteq \bigcap_{\alpha\in I} H_\alpha^{\geq}$. Let $\mathcal{R}(W)^I \subseteq \mathcal{R}(W)$ be the collection of all these chambers. The reflection arrangement $\mathcal{A}(W_I)$ is a subarrangement of $\mathcal{A}(W)$. Every chamber $C' \in \mathcal{R}(W_I)$ is the union of chambers of C of W with $C \subseteq C'$. The distinguished region $C_0 \in \mathcal{R}(W)$ determines a distinguished region $C_0' \in \mathcal{R}(W_I)$ and $\mathcal{R}(W)^I$ are precisely the regions of W that make up C_0'. Figure 5 illustrates the situation for $\mathfrak{S}_4$.

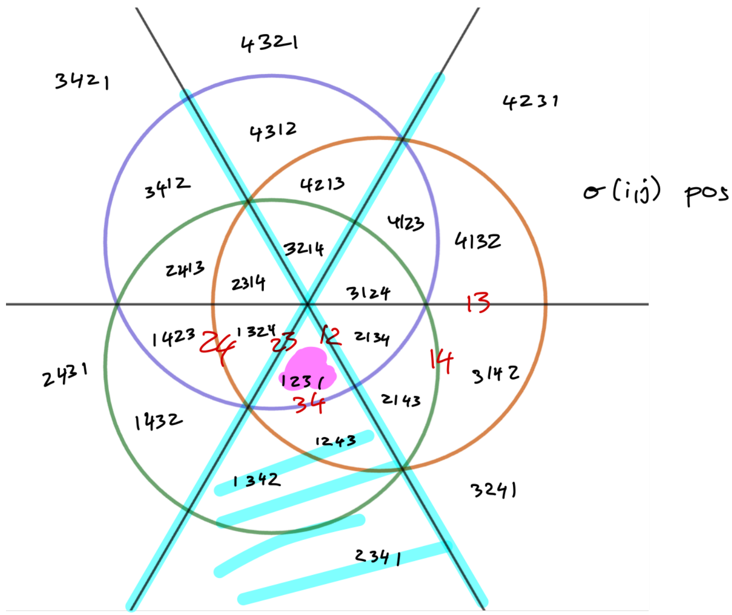


FIGURE 5. Stereographic projection of reflection arrangement of $\mathfrak{S}_4$. Permutations in black. The six reflections in red. Fundamental region in marker pink. Simple reflections for W_I in cyan. Black lines are reflection arrangement for W_I.

Whereas W_J is again a reflection group and we can hope to compute $W_J(q)$ inductively, the set W^J is not a group. We can, however, still compute $W^J(q)$ by inclusion-exclusion. For $w \in W$ define

$$D_R(w) \ := \ \{\alpha \in \Delta : \ell(ws_\alpha) < \ell(w)\} \,. \tag{2}$$

We call $\alpha \in D_R(w)$ a **(right) descent** of w. By the Exchange Condition of Corollary 4.10, $D_R(w)$ are precisely those s_α that occur as the final simple reflection in some reduced expression of w.

Example 5.8 (Type A_{n-1})**.** Recall that for $\sigma \in \mathfrak{S}_n$, $\ell(\sigma)$ is the number of inversions. For a simple reflection $s_i = (i, i+1)$, $\sigma \circ (i, i+1)$ exchanges $\sigma(i)$ and $\sigma(i+1)$ in the one-line notation. Thus, if $\ell(\sigma \circ (i, i+1)) < \ell(\sigma)$, then this means that $\sigma(i) > \sigma(i+1)$, that is, i is a *descent* as commonly defined for permutations. ◇

For $I \subseteq J \subseteq \Delta$, also define

$$\begin{aligned} D_I^J &:= \{w \in W : I \subseteq D_R(w) \subseteq J\} \\ D_I &:= D_I^I = \{w \in W : D_R(w) = I\} \\ W^J &= D_\varnothing^{\Delta \setminus J} \,. \end{aligned}$$

Note that the last line is *not* a definition but a straightforward consequence of the definition of W^J. Note also that

$$D_I^J \ = \ \biguplus_{I \subseteq K \subseteq J} D_K \quad \text{and thus} \quad D_I^J(q) \ = \ \sum_{I \subseteq K \subseteq J} D_K(q) \,.$$

Proposition 5.9. *For $I \subseteq J \subseteq \Delta$*

$$D_I^J(q) \ = \ \sum_{J \setminus I \subseteq K \subseteq J} (-1)^{|J \setminus K|} W^{\Delta \setminus K}(q) \,.$$

PROOF. Note that for any $K \subseteq J$, we can combine above insights to get

$$W^{\Delta \setminus K}(q) \ = \ D_\varnothing^K(q) \ = \ \sum_{L \subseteq K} D_L(q) \,.$$

Thus
$$\sum_{J\setminus I\subseteq K\subseteq J} (-1)^{|J\setminus K|} W^{\Delta\setminus K}(q) \;=\; \sum_{L\subseteq J} D_L(q) \sum_{(J\setminus I)\cup L\ \subseteq K\subseteq J} (-1)^{J\setminus K}\,.$$
Using the binomial theorem, we get that the last sum is 1 if and only $(J\setminus I)\cup L = J$, that is, $I \subseteq L \subseteq J$. Hence
$$\sum_{J\setminus I\subseteq K\subseteq J} (-1)^{|J\setminus K|} W^{\Delta\setminus K}(q) \;=\; \sum_{I\subset L\subseteq J} D_L(q) \;=\; D_I^J(q)\,. \qquad \square$$

For the next result we need the fact that relative to Δ, there is a unique element $w_0 \in W$ with $\ell(w_0) = |\mathcal{A}(W)|$. This is the **longest element** of Exercise 4.12.

Corollary 5.10.
$$\sum_{K\subseteq\Delta} \frac{(-1)^{|K|}}{W_K(q)} \;=\; \frac{q^{|\mathcal{A}|}}{W(q)}\,.$$
In particular for all $I\subseteq\Delta$
$$\sum_{K\subseteq\Delta} (-1)^{|K|}\frac{|W_I|}{|W_K|} \;=\; \frac{1}{|W^I|}\,.$$

Proof. Take $I = J = \Delta$. Then Proposition 5.9 and Corollary 5.7 yields
$$q^{|\mathcal{A}|} \;=\; D_\Delta^\Delta(q) \;=\; \sum_{K\subseteq\Delta}(-1)^{|\Delta\setminus K|}W^{\Delta\setminus K}(q) \;=\; \sum_{K\subseteq\Delta}(-1)^{|K|}W^{K}(q) \;=\; \sum_{K\subseteq\Delta}(-1)^{|K|}\frac{W(q)}{W_K(q)}\,. \qquad \square$$

For the symmetric group something amazing happens. If we compute the length generating function, we get
$$\mathfrak{S}_n(q) \;=\; [n]_q[n-1]_q\cdots[2]_q[1]_q \;=:\; [n]_q!\,. \tag{3}$$

Exercise 5.11. For a permutation $\tau\in\mathfrak{S}_n$ define $I(\tau) = (a_1,\dots,a_n)$ by $a_i := \#\{j > i : \tau(i) > \tau(j)\}$. This defines a map $I : \mathfrak{S}_n \to [n-1]\times[n-2]\times\cdots\times[2]\times[1]$.

(1) Show that I is injective. Hint: If $I(\sigma) = I(\tau)$, consider the maximal t with $\sigma^{-1}(t)\neq\tau^{-1}(t)$.
(2) Conclude that I is bijective.
(3) Prove (3) using I. ✎

This factorization into polynomials all whose coefficients are 1 happens for *all* finite reflection groups.

Theorem 5.12. *For any finite reflection group W there are positive integers $e_1, e_2, \dots, e_k$ such that*
$$W(q) \;=\; [e_1]_q[e_2]_q\cdots[e_k]_q\,.$$

6. The Coxeter complex

The reflection arrangement $\mathcal{A}(W)$ (and any arrangement, actually) induces a decomposition of the real vector space V, that we want to study now. Recall that a polyhedral cone is a set of the form $C = \{v\in V : \langle\beta_i, v\rangle \geq 0 \text{ for } i = 1,\dots,m\}$. A linear hyperplane $H_\alpha\subseteq V$ is **valid** for C if $C\subseteq H_\alpha^{\geq}$ and a **face** of C is a subset $F = C\cap H_\alpha$, where H_α is a valid hyperplane. Note that every face of a cone is again a cone. We also let $F = C$ be a face. The dimension of a cone is $\dim C = \dim \operatorname{span}(C)$.

For every $J\subseteq[m]$, the set
$$C_J := \{v\in C : \langle\beta_i, v\rangle = 0 \text{ for } i\in J\}$$
is a face of C with respect to the valid hyperplane H_β with $\beta = \sum_{i\in J}\beta_i$. Conversely, it can be shown that every face is of the form C_J for some J.

If C is a *simplicial* cone, that is, if $\beta_1,\dots,\beta_m$ are linearly independent, then $C_J = C_{J'}$ if and only if $J = J'$. Moreover, $\dim C_J = \dim V - |J|$. That is, the **codimension** of C_J is $\operatorname{codim} C_J = \dim V - \dim C_J = |J|$.

This gives us an easy way to count faces of a simplicial cone. The number of faces of codimension k is the coefficient of z^k of the polynomial

$$f_C(z) \;:=\; \sum_{J\subseteq[m]} z^{|J|} \;=\; \sum_{l=0}^{m}\binom{m}{l} z^l \;=\; (1+z)^m\,.$$

We'll see next why it is advantageous to use the codimension instead of dimension.

For our reflection arrangements $\mathcal{A}=\mathcal{A}(W)$, the space V is decomposed into cones of various dimensions. Let Φ_0^+ be a fixed positive system. For a point $p\in V$, define $\kappa_\alpha(p) := \operatorname{sgn}\langle\alpha,p\rangle \in \{-1,0,+1\}$ for each $\alpha\in\Phi_0^+$. Then

$$C_\kappa := \left\{ v\in V : \begin{array}{rcl} \langle\kappa_\alpha\alpha, v\rangle & \geq & 0 \text{ for } \kappa_\alpha\neq 0 \\ \langle\alpha, v\rangle & = & 0 \text{ for } \kappa_\alpha = 0 \end{array} \right\}$$

is a non-empty cone that contains p in its relative interior[1]; see Figure 6 for an example.

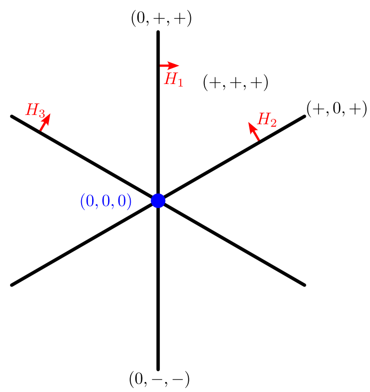


FIGURE 6. Decomposition of $\mathbb{R}^2$ into faces of various dimensions by $\mathcal{A}$. The f-polynomial is $f_{\mathcal{A}}(z) = 6+6z+z^2$.

Since all chambers of $\mathcal{A}$ are simplicial, the codimension of C_κ is $d(\kappa) := |\{\alpha\in\Phi^+ : \kappa_\alpha = 0\}|$. We define the **face vector** or **f-vector** $f(\mathcal{A})$ so that $f_k(\mathcal{A})$ is the number of cones C_κ of dimension k, for $k=0,1\,\ldots,n=\dim V$. We also define the **f-polynomial** as

$$f_{\mathcal{A}}(q) \;:=\; f_n(\mathcal{A}) + f_{n-1}(\mathcal{A})q + \cdots + f_0(\mathcal{A})q^n\,.$$

Now, if W is not essential, that is if if $V^W = \{v\in V : wv = v \text{ for all } w\in W\}\neq 0$. Then we can restrict the action of W to $U := (V^W)^\perp$. Thus W is an essential reflection group in U with reflection arrangement $\mathcal{A}|_U$. If F is a face of $\mathcal{A}$, then $F\cap U$ is a face of $\mathcal{A}|_U$. The faces change but the codimension is left untouched.

For a reflection arrangement $\mathcal{A}=\mathcal{A}(W)$, we can interpret $f_k(\mathcal{A})$ algebraically. Again, let $\Delta\subseteq\Phi^+\subset\Phi$ be fixed and let C be the fundamental domain with respect to Δ. Let W_J be a standard parabolic subgroup generated by s_α for $\alpha\in J\subseteq\Delta$. Recall that the rank of W_J is the codimension of the fixed space $V^{W_J} = L_J = \bigcap_{\alpha\in J} H_\alpha$, which is precisely $|J|$. Moreover J is a simple system for W_J, which we can identify with the face

$$C^J \;:=\; \{v : \langle\alpha,v\rangle = 0 \text{ for all } \alpha\in\Delta\setminus J, \langle\alpha,v\rangle\geq 0 \text{ for all } \alpha\in J\}\,,$$

which has codimension $|\Delta\setminus J|$. Note that wJ is a simple system for the parabolic subgroup wW_Jw^{-1} of the same rank and wJ is identified with wC^J. Let $\Phi_J^+ = \Phi^+\cap\operatorname{cone}(J)$, which is a positive system for W_J. Then

$$C^J \;=\; \bigcap_{\alpha\in Pos_J} H_\alpha^{\geq} \cap \bigcap_{\alpha\in\Phi^+\setminus Pos_J} H_\alpha\,.$$

[1]That is, the interior of C_κ relative to the linear subspace $\bigcap_{\kappa_\alpha=0} H_\alpha$.

Hence $wC^J = C^J$ if and only if $w \in W_{\Delta\setminus J}$ and hence wC^J for $w \in W^{\Delta\setminus J}$ represent the distinct simple systems for the parabolic subgroups that are conjugate to W_J, that is, wW_Jw^{-1} for $w \in W$. Therefore $(f_{\mathcal{A}(W)})_k$ is the number of simple systems of parabolic subgroups of rank k. For example for $k = n$, we have $J = \Delta$ and $W_J = W$. The number of simple systems is the number of regions of $\mathcal{A}(W)$, which is $W = W^{\varnothing}$. At the other extreme $k = 0$, we have $J = \varnothing$ and $W_J = \{e\}$. This group has only one simple system represented by $\bigcap \mathcal{A}(W)$ and $W^\Delta = \{e\}$. In summary

$$f_{\mathcal{A}(W)}(q) \;=\; \sum_{J\subseteq\Delta} |W^{\Delta\setminus J}| q^{|J|}\,.$$

Example 6.1 (Dihedral groups)**.** Let $\mathcal{D}_m$ be the dihedral symmetry group of an m-gon. The arrangement $\mathcal{A}_m = \mathcal{A}(\mathcal{D}_m)$ has m lines. The f-vector is then $f(\mathcal{A}_m) = (1, 2m, 2m)$. ⋄

Example 6.2 (Type A_{n-1})**.** For every $p \in \mathbb{R}^n$, there is a unique ordered set partition $M = (M_1, \dots, M_k)$ with $M_1 \cup \cdots \cup M_k = [n]$, $M_r \neq \varnothing$ and $M_r \cap M_s = \varnothing$ for all $r < s$. Such that

(1) $p_i = p_j$ if and only if $i, j \in M_l$ for some l;
(2) $p_i < p_j$ if and only if $i \in M_r$, $j \in M_s$ with $r < s$.

The cone containing p is precisely the set of $v \in V$ that give rise to the same set partition. The dimension of the cone is k, the length of the partition. The number of such *unordered* set partitions of n with k parts is counted by the famous **Stirling numbers** of the **second kind** $S(n, k)$. They are given by $S(0, 0) = 1$, $S(n, k) = 0$ for $k > n$ or $k = 0$ and

$$S(n, k) \;=\; kS(n-1, k) + S(n-1, k-1)\,.$$

Here is a table

$S(n,k)$	$k=1$	2	3	4	5	6
$n=1$	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

It follows that $f_k(\mathcal{A}) = k!S(n, k)$, as we have $k!$ ways to arrange the k parts of the partition. For example, for $n = 3$, we get $f(\mathcal{A}(\mathfrak{S}_3)) = (0, 1, 2! \cdot 3, 3! \cdot 1) = (0, 1, 6, 6)$, which is the f-vector for $\mathcal{D}_3$ but the triangle is embedded in 3-space. For $n = 4$, we get

$$f(\mathcal{A}(\mathfrak{S}_3)) \;=\; (0, 1, 14, 36, 24)\,.$$ ⋄

Exercise 6.3. Show the following identities of formal power series

(1) For fixed $k \geq 0$

$$\sum_{n\geq k} S(n, k)x^n \;=\; \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}\,.$$

(2) For $k \geq 0$

$$\sum_{n\geq k} S(n, k)\frac{x^n}{n!} \;=\; \frac{1}{k!}(e^x - 1)^k\,,$$

where $e^x = \sum_{h\geq 0} \frac{x^h}{h!}$. ✎

We will compute the f-vector or rather the f-polynomial of a simplicial arrangement $\mathcal{A}$ by a technique called **half-open** decomposition. Let C_0 be an arbitrary but fixed region. For any region $C \in \mathcal{R}(\mathcal{A})$ let $S(C)$ be the collection of hyperplanes $H \in \mathcal{A}$ that separate C from C_0. We define

$$\widehat{C} := C \setminus \bigcup_{H\in S(C)} H\,.$$

We call $\widehat{C}$ a **half-open cone**. Note that $\widehat{C}_0 = C_0$ and that the half-open cone of $-C_0$ is the interior of $-C_0$.

Lemma 6.4. *Let $\mathcal{A}$ be a hyperplane arrangement with regions $\mathcal{R}(\mathcal{A})$. Then*

$$V \;=\; \biguplus_{C\in\mathcal{R}(\mathcal{A})} \widehat{C}\,.$$

Proof. Let $p \in C_0^\circ$ be arbitrary but fixed. Let $v \in V$ be an arbitrary point. If $v \in C^\circ$ for some region C, then $\widehat{C}$ is unique region containing v. Otherwise, consider $q := v - \varepsilon(v-p)$ for $\varepsilon > 0$. For $\varepsilon > 0$ sufficiently small, q is contained in the interior of the unique $C \in \mathcal{R}(\mathcal{A})$ for which $v \in \partial C$. Let $H_\alpha \in \mathcal{A}$ be a hyperplane containing v and assume that $C_0 \subset H_\alpha^\geq$. Then $\langle \alpha, v\rangle = 0$ and, by construction, $\langle \alpha, q\rangle = \varepsilon\langle \alpha, p\rangle > 0$, since $p \in C_0^\circ$. Thus $C \subseteq H_\alpha^\geq$. This shows that $v \in \widehat{C}$.

Assume that C' is another region with $v \in \partial C'$. Then there is a wall H of C' separating C' from C. By construction of C, we have that q and and p lie on the same side, that is, H separates C' from C_0 and hence $v \notin \widehat{C'}$. □

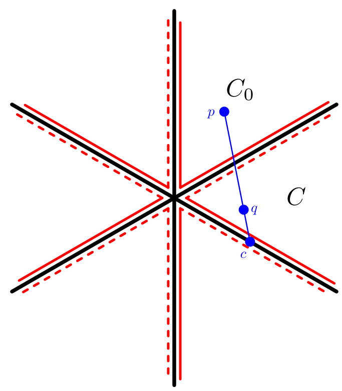


Figure 7. Half-open decomposition.

Let $C = \{v : \langle a_i, v\rangle \geq 0 \text{ for } i = 1,\ldots,n\}$ such that all H_{α_i} are walls of C. Let $D = D(C) := \{i : C_0 \subseteq H_{\alpha_i}^\leq\}$. Then

$$\widehat{C} \;=\; \{v : \langle \alpha_i, v\rangle \geq 0 \text{ for } i \notin D \text{ and } \langle \alpha_i, v\rangle > 0 \text{ for } i \in D\}\,.$$

The faces of $\widehat{C}$ are precisely those faces F of C that do not lie in any of the hyperplanes H_{α_i} for $i \in D$. If C is simplicial, then they are easy to count:

$$f_{\widehat{C}}(z) \;=\; \sum_{J\subseteq[m]\setminus D} z^{|J|} \;=\; (1-z)^{m-|D|}\,.$$

Note that if $0 \leq |D(C)| \leq m$ and we define the h-**polynomial**

$$h_{\mathcal{A}}(z) \;=\; \sum_{C\in\mathcal{R}(\mathcal{A})} z^{m-|D(C)|} \;=\; h_m(\mathcal{A}) + h_{m-1}(\mathcal{A})z + \cdots + h_0(\mathcal{A})z^m\,.$$

Theorem 6.5. *Let $\mathcal{A}$ be a simplicial hyperplane arrangement with base region $C_0 \in \mathcal{R}(\mathcal{A})$ and h-polynomial $h_{\mathcal{A}}(z)$. Then*

$$f_{\mathcal{A}}(z) \;=\; h_{\mathcal{A}}(z+1)$$

Proof. It follows from Lemma 6.4 that for every face F of $\mathcal{A}$, there is a unique $C \in \mathcal{R}(\mathcal{A})$ with $F \subseteq \widehat{C}$. Thus

$$f_{\mathcal{A}}(z) \;=\; \sum_{C\in\mathcal{R}(\mathcal{A})} f_{\widehat{C}}(z) \;=\; \sum_{C\in\mathcal{R}(\mathcal{A})} (1+z)^{m-|D(C)|} \;=\; \sum_{k=0}^{m} h_k(\mathcal{A})(1+z)^{m-k}\,.$$ □

Our notation does not emphasize which base region we selected for the computation of the half-open decomposition or $h_{\mathcal{A}}(z)$. But it does not matter.

Corollary 6.6. *The h-polynomial is independent of the base region C_0.*

PROOF. Note that $h_{\mathcal{A}}(z) = f_{\mathcal{A}}(z-1)$. □

Corollary 6.7 (Dehn–Sommerville equations)**.** *If $\mathcal{A}$ is an essential simplicial arrangement in n-dimensional space, then $h_{\mathcal{A}}(z) = z^n h_{\mathcal{A}}(\frac{1}{z})$, that is,*

$$h_k(\mathcal{A}) \;=\; h_{n-k}(\mathcal{A})$$

PROOF. A wall of C separates C from C_0 if and only if it does not separate C from $-C_0$. Thus, the h-polynomial with respect to $-C_0$ as the base region is precisely

$$\sum_{C\in\mathcal{R}(\mathcal{A})} z^{|D(C)|} \;=\; z^n h_{\mathcal{A}}(\tfrac{1}{z})\,.$$

Since the $h_{\mathcal{A}}(z)$ is independent of the choice of a base region now complete the proof. □

Note that we can express the h-polynomial of a reflection arrangement $\mathcal{A}(W)$ algebraically as follows. Recall from (2) that $\alpha \in \Delta$ is a **right descent** of $w \in W$ if $\ell(ws_\alpha) < \ell(w)$ and $D_R(w)$ is the **descent set** of w. By the Exchange condition of Corollary 4.10, $\alpha \in D_R(w)$ if and only if there is a reduced expression that ends in s_α. Using Construction 4.5, this means that $t := ws_\alpha w^{-1}$ is a reflection in a wall of wC_0 that separates wC_0 from C_0. That is

$$|D(wC_0)| \;=\; |D_R(w)|\,.$$

We write $d(w) := |D_R(w)|$ and define the **Eulerian polynomial**

$$A_W(z) \;:=\; \sum_{w\in W} z^{d(w)}\,.$$

The naming is inspired by the **Eulerian numbers** $A(n,k)$ that count the number of permutations $\sigma \in \mathfrak{S}_n$ with k descents, that is, indices $1 \le i < n$ with $\sigma(i) > \sigma(i+1)$. For example

$$A_{\mathfrak{S}_3}(z) = 1 + 4z + z^2 \quad A_{\mathfrak{S}_4}(z) = 1 + 11z + 11z^2 + 1\,;$$

see Figure 8.

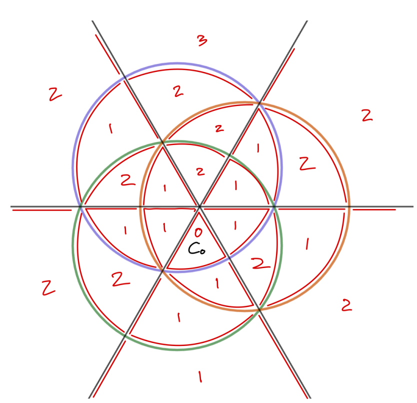


FIGURE 8. Half-open decomposition for reflection arrangement of symmetric group

7. Classification of finite reflection groups

In this section we will classify all the finite reflection groups. For that we will first seek a compact way to encode reflection groups. Let W be a reflection group that acts essentially on the n-dimensional real vector space V. Let Φ be a root system consisting of vectors of unit length and let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be simple system. Remember that $s_{\alpha_i}, s_{\alpha_j}$ for $i \neq j$ generates a rank-2 reflection group $\mathcal{D}_{m_{ij}}$ for some $m_{ij} \geq 2$. Since α_i, α_j is a simple system for $\mathcal{D}_{m_{ij}}$, we have that $s_{\alpha_i} s_{\alpha_j}$ is a rotation by $\frac{2\pi}{m_{ij}}$. Hence $(s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = e$ and

$$\langle \alpha_i, \alpha_j \rangle \;=\; -\cos \frac{\pi}{m_{ij}} \,.$$

We define the **Coxeter matrix** $M = (m_{ij})_{i,j}$. Let denote the Gram matrix of the simple system by A with entries

$$A_{ij} \;=\; \langle \alpha_i, \alpha_j \rangle \;=\; -\cos \frac{\pi}{m_{ij}} \,, \tag{4}$$

where $m_{ii} = 1$ and $A_{ii} = 1$. Up to labelling the individual roots, the matrix A is independent of the choice of a simple system. It is more customary to record M by defining the **Coxeter graph** Γ, which is a simple undirected graph on nodes $[n] = \{1, \dots, n\}$ such that ij is an edge iff $m_{ij} \geq 2$. The edge ij carries an edge label m_{ij} iff $m_{ij} \geq 3$.

We call W **irreducible** if the Coxeter graph is connected. For example the Coxeter graph of the dihedral group $I_2(2)$ consists on two isolated nodes. If $V_1 \uplus V_2 \uplus \cdots \uplus V_k = [n]$ are the node sets of the connected components of Γ, then $W_h = \langle s_i : i \in V_h \rangle$ defines a (standard) parabolic subgroup and $W = W_1 \times W_2 \times \cdots \times W_n$. Hence, it suffices to classify the irreduce reflection groups. Figure 9 shows the Coxeter graphs for the dihedral groups as well as the groups of types A_n and B_n and some more.

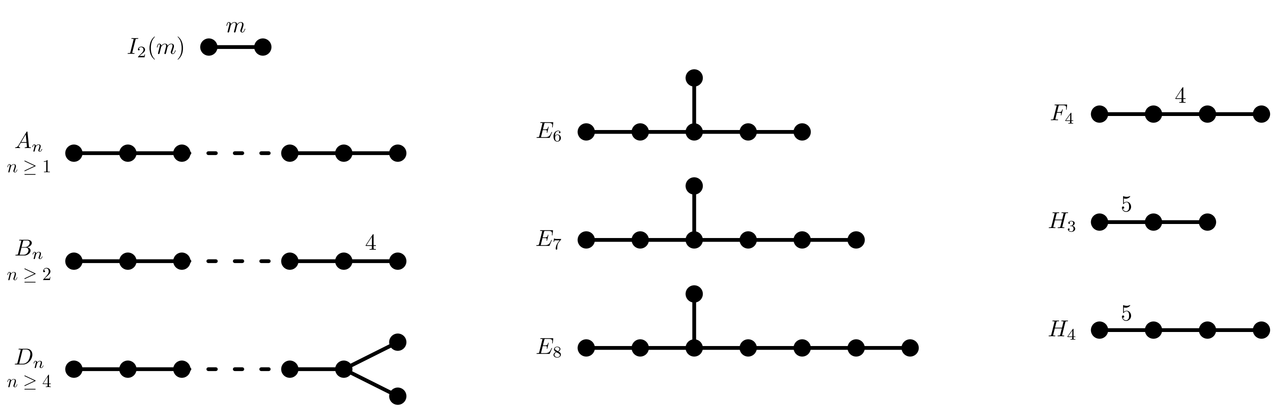


FIGURE 9. Coxeter graphs of all reflection groups.

The Coxeter graph is connected if and only if the Gram matrix A is **indecomposable**, that is, if there is no partition $I \uplus J = [n]$ such that $A_{ij} = 0$ for all $i \in I, j \in J$. Moreover, as a Gram matrix, A is symmetric and **positive semidefinite**, i.e., $x^t A x \geq 0$ for all $x \in V$. Moreover, A is **positve definite** if additionally $x^t A x = 0$ implies $x = 0$.[2] We need the following lemma from matrix theory.

Lemma 7.1. *Let A be an indecomposable and positive semidefinite matrix such that $A_{ij} \leq 0$ for all $i \neq j$.*

a) $\ker A = \{x : x^t A x = 0\}$
b) $\dim \ker A \leq 1$ *and if* $= 1$*, then* $\ker(A)$ *is spanned by a vector with all entries positive.*
c) *The eigenspace for the smallest eigenvalue is* 1*-dimensional an spanned by vector with strictly positive components.*

[2] Yes, we should consistently write $\langle x, Ax \rangle$ but this is too much.

Proof. a) Since A is positive definite and hence the Gram matrix of some vector configuration, there is G with $A = G^tG$ and hence $x^tAx = \|Gx\|^2$. Thus, if $x^tAx = 0$, then $Gx = 0$ and hence $Ax = 0$. The direction $Ax = 0$ implies $x^tAx = 0$ is clear. (Here we used the positive semidefiniteness.)

b) Let $x \in \ker A$ with $x \neq 0$. Define z by $z_i = |x_i|$. Then

$$0 \;\leq\; z^tAz \;=\; \sum_{i,j} |x_i|A_{ij}|x_j| \;\leq\; \sum_{i,j} x_iA_{ij}x_j \;=\; 0\,,$$

where we used $A_{ij} \leq 0$, we conclude that $Az = 0$. Define $I := \{i : z_i = 0\}$ and $J := [n] \setminus I$. For every i, we have

$$0 \;=\; \sum_{j \in J} A_{ij}z_j\,.$$

Again, since $A_{ij} \leq 0$ and $z_j > 0$ for $j \in J$, we get $A_{ij} = 0$ for all $i \in I$ and $j \in J$. Since A is assumed to be irreducile, this means $I = \varnothing$. This shows that $z > 0$. In fact the argument shows that every $x \neq 0$ with $Ax = 0$ has to have all coordinates nonzero. However if there is $x' \in \ker A$ linearly independent from x, then $\alpha x + \beta x'$ has a zero component for some $\alpha, \beta \in \mathbb{R}$. This shows b).

For c), we note that if $\lambda \geq 0$ is the smallest eigenvalue, then $A' := A - \lambda \mathrm{I}$ is still positive semidefinite, irreducile and non-positive off-diagonal entries. The kernel of A' is the eigenspace of A and is 1-dimensional by a). ☐

Every graph Γ on nodes $[n]$ with edge labels $m_{ij} \geq 3$ gives rise to a matrix A_Γ as defined (4). We call Γ positive (semi)definite if A_Γ is positive (semi)definite. A **subgraph** Γ' is obtained from Γ by deleting nodes together with incident edges or decreasing the label on some edges.

Proposition 7.2. *Let Γ be a connected, positive definite graph Γ, then every subgraph Γ' is positive definite.*

Proof. Suppose first that Γ' is obtained by only deleting nodes and let $J \subsetneq [n]$ be the node set of Γ'. The matrix $A_{\Gamma'}$ is obtained from A_Γ by restricting to rows and columns indexed by J. In particular, if $A_{\Gamma'}$ is not of full rank, then by Lemma 7.1a), there is an x with $\mathrm{supp}(x) = \{i : x_i \neq 0\} \subseteq J$ and $x^tA_\Gamma x = 0$. But if $x \neq 0$, then it follows from Lemma 7.1b) that $J = [n]$, which shows $\Gamma = \Gamma'$ and Γ was positive definite.

Thus, it suffices to assume that Γ' has the same node set but $m'_{ij} \leq m_{ij}$ for all edges ij. That is, $A_{ij} = -\cos(\frac{\pi}{m_{ij}}) \leq -\cos(\frac{\pi}{m'_{ij}}) = A'_{ij}$. Now if there is $x \neq 0$ with $x^tA'x \leq 0$, then we compute

$$0 \;\leq\; \sum_{i,j} a_{ij}|x_i||x_j| \;\leq\; \sum_{i,j} a'_{ij}|x_i||x_j| \;\leq\; \sum_{i,j} a'_{ij}x_ix_j \;\leq\; 0\,.$$

The vector z of absolute values of x thus satisfies $z^tAz = 0$, which contradicts our assumption that Γ and hence $A = A_\Gamma$ is positive definite. ☐

Theorem 7.3. *The only connected, positive definite graphs are those in Figure 9.*

Proof. **1. Γ is a tree (i.e. does not contain a cycle).**
The matrix A of a cycle with all edge labels $m_{ij} = 3$ consists of cyclic shifts of the row $(\rho, 1, \rho, 0, \ldots, 0)$, where $\rho = -\cos(\frac{\pi}{3}) = -\frac{1}{2}$. Hence $\mathbf{1}^tA\mathbf{1} = 0$ and a cycle cannot occur as a subgraph.

This means that Γ is a tree.

If Γ is not of type A_n, then it has node of degree ≥ 3.

2. Γ has no node of degree ≥ 4.
Otherwise, we could find as a subgraph, the graph Γ' with nodes $1, 2, 3, 4, 5$ and edges $i5$ for $i = 1, \ldots, 4$.

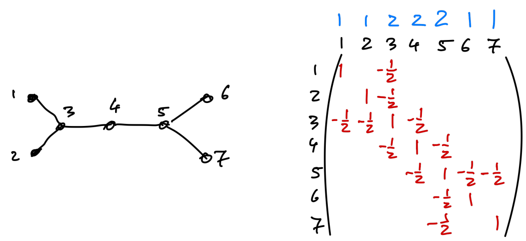
The corresponding matrix $A_{\Gamma'}$ is

$$\begin{pmatrix} 1 & & & & -\frac{1}{2} \\ & 1 & & & -\frac{1}{2} \\ & & 1 & & -\frac{1}{2} \\ & & & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

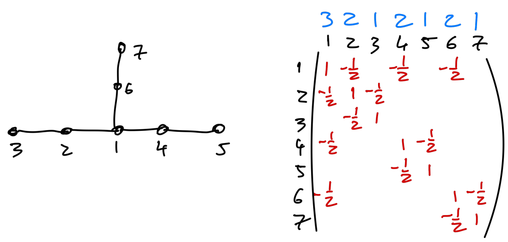
which is not positive definite.

2. Γ has a unique node of degree $= 3$

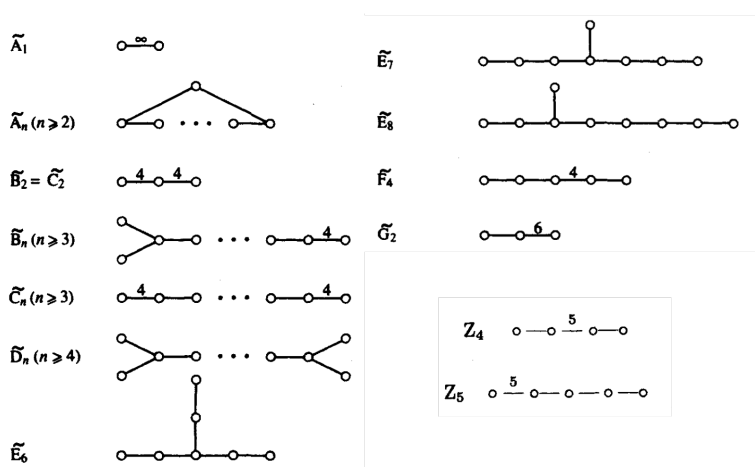
The following illustrates the situation and hints at a generalization



3. The three paths starting at the branch point cannot be all ≥ 2 long. The following illustrates the situation



In a similar manner, one checks that the following subgraphs cannot occur[3] :



□

This shows that if W is a finite reflection group, then its Coxeter graph has to belong to one of the 10 types shown in Figure 9. To construct a finite reflection group for each of the 10 types, let Γ be one of these on n nodes. The matrix $A = A_\Gamma \in \mathbb{R}^{n\times n}$ is positive definite and there is a factorization $A = G^t G$ for some full-rank matrix $G \in \mathbb{R}^{n\times n}$. Let $\alpha_1, \ldots, \alpha_n$ be the column vectors. Note that $A_{ii} = 1$ and all

[3]The figure is taken from [?]

α_i are unit vectors. Define hyperplanes $H_i = H_{\alpha_i}$ and reflections $s_i = s_{\alpha_i}$ and let W be the subgroup of $\mathrm{GL}(\mathbb{R}^n)$ generated by these reflections. Using the fact that $\langle \alpha_i, \alpha_j \rangle \leq 0$ for all $i \neq j$ we can employ ideas similar to those of Section 3 to show that for every α_i, the orbit $W\alpha_i$ is finite. If we let Φ be the union of all these finite orbits, then this allows us to show that $W \to \mathfrak{S}(\Phi)$ is a monomorphism and hence W is finite; cf. Exercise 2.3.

Theorem 7.4. *The irreducible finite reflection groups are, up to isomorphism, in one-to-one correspondence with the Coxeter graphs of Figure 9.*

CHAPTER 2

Arrangements and simpliciality

The beginning of this chapter essentially follows the first chapters of the book *Arrangements of Hyperplanes* by Orlik and Terao [?].

1. Arrangements of hyperplanes

Definition 1.1. Let K be a field, $\ell \in \mathbb{N}$, and $V := K^\ell$. An **arrangement of hyperplanes** (or **ℓ-arrangement**) $(\mathcal{A}, V)$ (or $\mathcal{A}$ for short) is a finite set of hyperplanes $\mathcal{A}$ in V.

Example 1.2. Let $K = \mathbb{F}_q$ be a finite field and $V = \mathbb{F}_q^\ell$. The set of all linear hyperplanes in V is an arrangement of hyperplanes.

Remark 1.3. Let $e_1, \dots, e_\ell$ be a basis of V and $x_1, \dots, x_\ell \in V^*$ be the dual basis. The symmetric algebra $S(V^*)$ is a polynomial algebra $K[x_1, \dots, x_\ell]$.

Each hyperplane H in an arrangement is the kernel of a linear form α_H (= polynomial of degree 1 in $S(V^*)$). However, for any unit $a \in K^\times$, the linear form $a \cdot \alpha_H$ defines the same hyperplane. So in a certain sense, an arrangement of hyperplanes may equivalently be defined as a set of points in the projective space $\mathbb{P}(V^*)$.

Since α_H is defined only up to scalars, it will be convenient to write

$$f \doteq g \quad \Leftrightarrow \quad \exists a \in K^\times \,:\, f = a \cdot g.$$

Definition 1.4. Let $\mathcal{A}$ be an arrangement and write α_H, $H \in \mathcal{A}$ for defining linear forms. The product

$$Q(\mathcal{A}) \doteq \prod_{H \in \mathcal{A}} \alpha_H$$

is called the **defining polynomial** of $\mathcal{A}$.

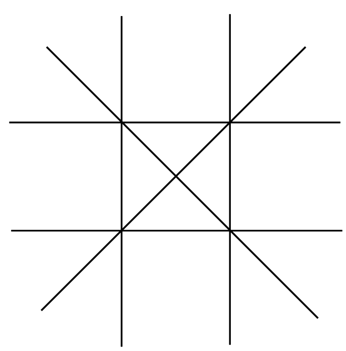


FIGURE 1. The arrangement of type A_3.

Example 1.5. The arrangement $\mathcal{A}$ of type A_3 has 6 hyperplanes in $V = \mathbb{R}^3$ (see Figure 1). With respect to the standard basis x, y, z of V^*, the hyperplanes of $\mathcal{A}$ are the kernels of the linear forms

$$(1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,1,1).$$

We have $S(V^*) \cong K[x, y, z]$ and

$$Q(\mathcal{A}) = xyz(x+y)(y+z)(x+y+z).$$

Example 1.6. The arrangement in $V = K^\ell$ defined by

$$Q(\mathcal{A}) = \prod_{i=1}^{\ell} x_i$$

is called the **boolean arrangement**.

Example 1.7. For $1 \leq i < j \leq \ell$ let $H_{i,j} = \ker(x_i - x_j)$. The arrangement in $V = \mathbb{R}^\ell$ defined by

$$Q(\mathcal{A}) = \prod_{1 \leq i < j \leq \ell} (x_i - x_j)$$

consisting of all the hyperplanes $H_{i,j}$ is called the **braid arrangement**.

Exercise 1.8. Compute the number $|\mathcal{A}|$ of hyperplanes for the arrangements in examples 1.6, 1.7, 1.2.

Definition 1.9. Let $\mathcal{A}$ be an arrangement. The set

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in U} H \mid U \subseteq \mathcal{A} \right\}$$

is called the **intersection lattice** of $\mathcal{A}$. It is partially ordered by reverse inclusion:

$$X \leq Y \iff Y \subseteq X, \quad \text{for } X, Y \in L(\mathcal{A}).$$

If $X \in L(\mathcal{A})$, then the **rank** $r(X)$ of X is defined as $r(X) := \ell - \dim X$, i.e. the codimension of X and the rank of the arrangement $\mathcal{A}$ is defined as $r(\mathcal{A}) := r(T(\mathcal{A}))$ where $T(\mathcal{A}) := \bigcap_{H \in \mathcal{A}} H$ is the **center** of $\mathcal{A}$.
The arrangement $\mathcal{A}$ is called **central** if $0 \in H$ for all $H \in \mathcal{A}$.
An ℓ-arrangement $\mathcal{A}$ is called **essential** if $r(\mathcal{A}) = \ell$.

Example 1.10. The braid arrangement $\mathcal{A}$ is not essential:

$$T(\mathcal{A}) = \langle (1, \ldots, 1) \rangle, \quad r(\mathcal{A}) = \ell - \dim(T(\mathcal{A})) = \ell - 1.$$

Example 1.11. Recall the arrangement $\mathcal{A}$ of type A_3. It is essential and of rank 3. The intersection lattice consists of 0, V, $H \in \mathcal{A}$ and the subspaces generated by

$$(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, -1), (1, 0, -1), (1, -1, 0), (1, -1, 1).$$

Definition 1.12. Let $\mathcal{A}$ be an arrangement. For $X \in L(\mathcal{A})$, we define the **localization**

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$$

of $\mathcal{A}$ at X, and the **restriction of** $\mathcal{A}$ to X, $(\mathcal{A}^X, X)$, where

$$\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

Example 1.13. Localizations and restrictions in the arrangement of type A_3 (Picture).

Definition 1.14 ([?, 2.13, 2.15])**.** Let $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ be arrangements. The **product** of $\mathcal{A}_1$ and $\mathcal{A}_2$ is the arrangement $(\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$ where

$$\mathcal{A}_1 \times \mathcal{A}_2 := \{H \oplus V_2 \mid H \in \mathcal{A}_1\} \cup \{V_1 \oplus H \mid H \in \mathcal{A}_2\}.$$

An arrangement $(\mathcal{A}, V)$ is called **reducible** if there exist arrangements $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ such that $(\mathcal{A}, V) = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$. Otherwise $(\mathcal{A}, V)$ is called **irreducible**.

Example 1.15. The arrangement in Figure 2 is called a **near pencil**. It is reducible.

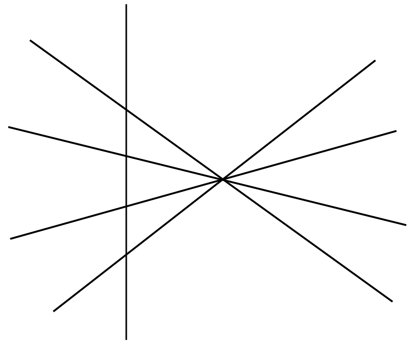


FIGURE 2. A near pencil arrangement.

2. Simplicial arrangements

Definition 2.1. Let $\ell \in \mathbb{N}$, $V := \mathbb{R}^\ell$, and $\mathcal{A}$ an arrangement in V. Let $\mathcal{K}(\mathcal{A})$ be the set of connected components (**chambers**) of $V \setminus \bigcup_{H\in\mathcal{A}} H$. If every chamber K is an **open simplicial cone**, i.e. there exist $\alpha_1^\vee, \ldots, \alpha_\ell^\vee \in V$ such that

$$K = \Big\{ \sum_{i=1}^{\ell} a_i \alpha_i^\vee \mid a_i > 0 \quad \text{for all} \quad i = 1, \ldots, \ell \Big\} =: \langle \alpha_1^\vee, \ldots, \alpha_\ell^\vee \rangle_{>0},$$

then $\mathcal{A}$ is called a **simplicial arrangement**.

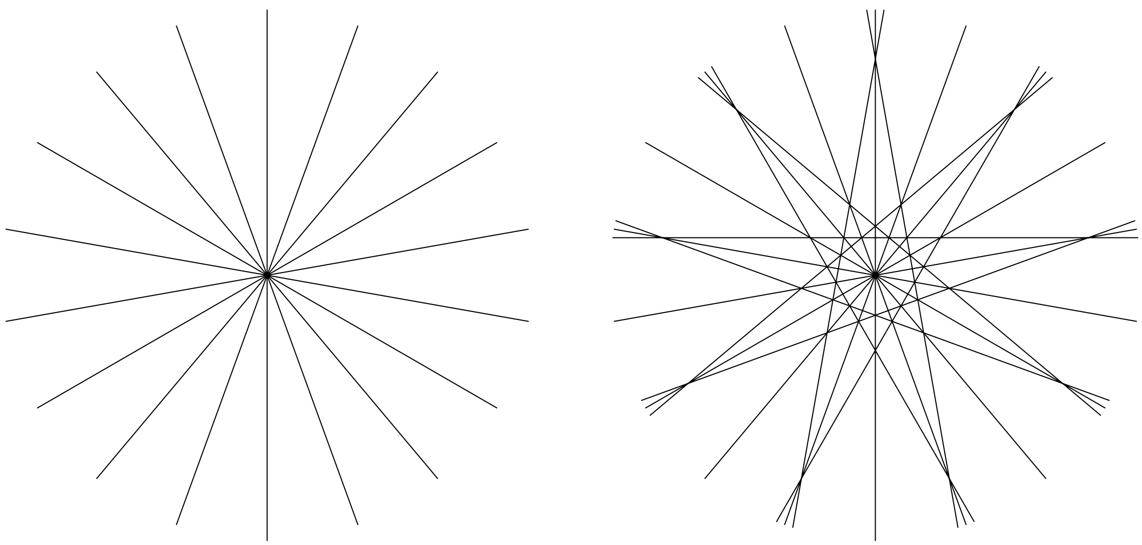


FIGURE 3. A simplicial arrangement in $\mathbb{R}^2$, a representation of a simplicial arrangement in $\mathbb{R}^3$ in the projective plane.

Example 2.2. (1) Figure 3 displays examples for $\ell = 2$ and $\ell = 3$.
(2) Let W be a real reflection group, $R \subseteq V^*$ the set of roots of W. For $\alpha \in V^*$ we write $\alpha^\perp = \ker(\alpha)$. Then $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ is a simplicial arrangement.

Theorem 2.3 (Deligne, 1972)**.** The complement of a complexified finite simplicial arrangement is $K(\pi, 1)$.

For dimension three, there is a catalogue of known simplicial arrangements [**?**] and we have a complete list of simplicial arrangements with at most 27 lines [**?**]. There are a little less than 100 "sporadic" arrangements in the catalogue. For most of them, we have no satisfactory explanation yet. H.S.M. Coxeter writes

"[...] the diagrams which profess to portray these known polygrams are strangely unintelligible."

3. Characteristic polynomial and deletion-restriction

Definition 3.1. We write Φ_ℓ for the **empty arrangement** in $V = K^\ell$.

Definition 3.2 ([?, 1.13, 2.25])**.** Let $\mathcal{A}$ be an arrangement in $V = K^\ell$. The **Möbius function** of $L(\mathcal{A})$ is the map $\mu : L(\mathcal{A}) \to \mathbb{Z}$ defined recursively by

$$\mu(V) = 1, \quad \sum_{Z \leq Y} \mu(Z) = 0 \quad \text{if } V < Y \in L(\mathcal{A}).$$

The **Poincaré polynomial** $\pi_\mathcal{A} \in \mathbb{Z}[t]$ of $\mathcal{A}$ is defined by

$$\pi_\mathcal{A}(t) = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{r(X)}.$$

The **characteristic polynomial** $\chi_\mathcal{A} \in \mathbb{Z}[t]$ of $\mathcal{A}$ is defined by

$$\chi_\mathcal{A}(t) = t^\ell \pi_\mathcal{A}(-t^{-1}) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim(X)}.$$

Example 3.3. Let $\mathcal{A}$ be an arrangement in V. Then $\mu(V) = 1$, $\mu(H) = -1$ for all $H \in L(\mathcal{A})$. If $r(X) = 2$, $X \in L(\mathcal{A})$, then $\mu(X) = |\mathcal{A}_X| - 1$.

Example 3.4. Möbius function of the arrangement $\mathcal{A}$ of type A_3. The characteristic polynomial is $\chi_\mathcal{A}(t) = (t-1)(t-2)(t-3)$, the Poincaré polynomial is $\pi_\mathcal{A}(t) = (1+t)(1+2t)(1+3t)$.

Exercise 3.1. Let $\mathcal{A}$ be the boolean arrangement given by $Q(\mathcal{A}) = x_1 \cdots x_\ell$. Prove that $\mu(X) = (-1)^{r(X)}$ for $X \in L(\mathcal{A})$. ✎

Definition 3.5. Let $\mathcal{A}$ be an arrangement in V and $H \in \mathcal{A}$. Then $(\mathcal{A}, \mathcal{A}' = \mathcal{A}\backslash\{H\}, \mathcal{A}'' = \mathcal{A}^H)$ is called a **triple of arrangements** with respect to H.

Lemma 3.6. *Let $\mathcal{A}$ be a central arrangement. Then*

$$\pi_\mathcal{A}(t) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|}(-t)^{r(\mathcal{B})}.$$

Proof. Let $X \in L(\mathcal{A})$. We write

$$S(X) := \{\mathcal{B} \subseteq \mathcal{A} \mid T(\mathcal{B}) = X\}, \quad \nu(X) := \sum_{\mathcal{B} \in S(X)} (-1)^{|\mathcal{B}|}.$$

Note that we have a partition

$$\{\mathcal{B} \subseteq \mathcal{A} \mid \mathcal{B} \subseteq \mathcal{A}_X\} = \dot{\bigcup}_{Z \leq X} S(Z).$$

Thus $\nu(V) = (-1)^{|\varnothing|} = 1$, and if $V < X$,

$$\sum_{Z \leq X} \nu(Z) = \sum_{Z \leq X} \sum_{\mathcal{B} \in S(Z)} (-1)^{|\mathcal{B}|} = \sum_{\mathcal{B} \subseteq \mathcal{A}_X} (-1)^{|\mathcal{B}|} = 0$$

since $\mathcal{A}_X \neq \varnothing$. But this means that ν satisfies the same recursion as μ, hence $\nu = \mu$. Now

$$\pi_\mathcal{A}(t) = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{r(X)} = \sum_{X \in L(\mathcal{A})} \sum_{\mathcal{B} \in S(X)} (-1)^{|\mathcal{B}|}(-t)^{r(X)}.$$

If $\mathcal{B} \in S(X)$, then $r(\mathcal{B}) = r(X)$ since $T(\mathcal{B}) = X$. Since every $\mathcal{B} \subseteq \mathcal{A}$ occurs in a unique $S(X)$, we obtain the claimed formula. □

Theorem 3.7 (**Deletion-Restriction**)**.** *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements. Then*

$$\pi_\mathcal{A}(t) = \pi_{\mathcal{A}'}(t) + t\pi_{\mathcal{A}''}(t), \quad \chi_\mathcal{A}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t).$$

Proof. (compare [?, Thm. 2.56]) Let H be the distinguished hyperplane in the triple; we use Lemma 3.6:

$$\begin{aligned}\pi_{\mathcal{A}}(t) &= \sum_{\mathcal{B}\subseteq\mathcal{A}}(-1)^{|\mathcal{B}|}(-t)^{r(\mathcal{B})} = \sum_{H\notin\mathcal{B}\subseteq\mathcal{A}}(-1)^{|\mathcal{B}|}(-t)^{r(\mathcal{B})} + \sum_{H\in\mathcal{B}\subseteq\mathcal{A}}(-1)^{|\mathcal{B}|}(-t)^{r(\mathcal{B})}\\ &= \pi_{\mathcal{A}'}(t) + \sum_{H\in\mathcal{B}\subseteq\mathcal{A}}(-1)^{|\mathcal{B}|}(-t)^{r(\mathcal{B})}.\end{aligned}$$

For the second summand, we apply the proof of Lemma 3.6 to the arrangement $\mathcal{A}''$ in the vector space H: We write $S''(Y) = \{\mathcal{B}\subseteq\mathcal{A} \mid T(\mathcal{B}) = Y,\ H\in\mathcal{B}\}$ for $Y\in L(\mathcal{A}'')$. Then

$$\begin{aligned}\pi_{\mathcal{A}}(t) - \pi_{\mathcal{A}'}(t) &= \sum_{H\in\mathcal{B}\subseteq\mathcal{A}}(-1)^{|\mathcal{B}|}(-t)^{r(\mathcal{B})}\\ &= \sum_{Y\in L(\mathcal{A}'')}\sum_{\mathcal{B}\in S''(Y)}(-1)^{|\mathcal{B}|}(-t)^{r(Y)}\\ &= \sum_{Y\in L(\mathcal{A}'')}\sum_{\mathcal{B}\in S''(Y)}(-1)(-1)^{|\mathcal{B}\setminus\{H\}|}(-t)(-t)^{r(Y)-1}\\ &= t\sum_{Y\in L(\mathcal{A}'')}\sum_{\mathcal{B}\in S''(Y)}(-1)^{|\mathcal{B}\setminus\{H\}|}(-t)^{r(Y)-1}\\ &= t\pi_{\mathcal{A}''}(t)\end{aligned}$$

where the last equality is Lemma 3.6 for the arrangement $\mathcal{A}''$ (note that the rank decreases by one in the restriction because $\dim V = 1+\dim H$ and hence $r(Y) = r''(Y)+1$):

$$\sum_{\mathcal{B}\in S''(Y)}(-1)^{|\mathcal{B}\setminus\{H\}|} = \sum_{\mathcal{B}''\subseteq\mathcal{A}'',\,T(\mathcal{B}'')=Y}\ \sum_{H\in\mathcal{B}\subseteq\mathcal{A},\,\mathcal{B}^H=\mathcal{B}''}(-1)^{|\mathcal{B}\setminus\{H\}|} = \sum_{\mathcal{B}''\subseteq\mathcal{A}'',\,T(\mathcal{B}'')=Y}(-1)^{|\mathcal{B}''|}$$

by Exercise 3.2. The recursion for the characteristic polynomial immediately follows from the one for the Poincaré polynomial. □

Exercise 3.2. Let $\mathcal{A}$ be an arrangement, $H_0\in\mathcal{A}$, and $\mathcal{A}'' := \mathcal{A}^{H_0}$ be the restriction of $\mathcal{A}$ to H_0. Show that for $\mathcal{B}''\subseteq\mathcal{A}''$,

$$(-1)^{|\mathcal{B}''|} + \sum_{H_0\in\mathcal{B}\subseteq\mathcal{A},\,\mathcal{B}^{H_0}=\mathcal{B}''}(-1)^{|\mathcal{B}|} = 0.$$

✎

Theorem 3.8 (Zaslavsky, 1975)**.** Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^r$. Then

$$|\mathcal{K}(\mathcal{A})| = (-1)^r\chi_{\mathcal{A}}(-1).$$

Proof. Let $(\mathcal{A},\mathcal{A}',\mathcal{A}'')$ be a triple of arrangements with respect to H. Denote P the set of those chambers in $\mathcal{K}(\mathcal{A}')$ which intersect H, and Q the set of those chambers in $\mathcal{K}(\mathcal{A}')$ which do not intersect H, so $|\mathcal{K}(\mathcal{A}')| = |P|+|Q|$. Since H divides each chamber in P into two chambers of $\mathcal{A}$, $|\mathcal{K}(\mathcal{A})| = 2|P|+|Q|$. But $P\to\mathcal{K}(\mathcal{A}'')$, $C\mapsto C\cap H$ is a bijection, hence $|\mathcal{K}(\mathcal{A}'')| = |P|$. This proves

$$|\mathcal{K}(\mathcal{A})| = |\mathcal{K}(\mathcal{A}')| + |\mathcal{K}(\mathcal{A}'')|. \tag{5}$$

Now if $\mathcal{A}$ is the empty arrangement, then $|\mathcal{K}(\mathcal{A})| = 1 = (-1)^r\chi_{\mathcal{A}}(-1)$ since $\chi_{\mathcal{A}} = t^r$. If $\mathcal{A}\neq\varnothing$, then we obtain the claim using induction, (5), and deletion-restriction (3.7). □

Theorem 3.9 ([?, Thm. 2.69])**.** Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_q^\ell$ and $M(\mathcal{A})$ be the complement of the union of all hyperplanes in $\mathcal{A}$. Then

$$|M(\mathcal{A})| = \chi_{\mathcal{A}}(q).$$

Proof. We have $|M(\Phi_\ell)| = q^\ell = \chi_{\Phi_\ell}(q)$. If $\mathcal{A}\neq\varnothing$, let $(\mathcal{A},\mathcal{A}',\mathcal{A}'')$ be a triple. Then $|M(\mathcal{A})| = |M(\mathcal{A}')| - |M(\mathcal{A}'')|$, thus $|M(\mathcal{A})|$ and $\chi_{\mathcal{A}}(q)$ satisfy the same recursion (Thm. 3.7) and agree on Φ_ℓ. □

Exercise 3.3. Compute $\chi_{\mathcal{A}}(q)$ for the arrangement in $\mathbb{F}_q^\ell$ defined by

$$Q(\mathcal{A}) = \prod_{1\le i\le j\le \ell} (x_i + \ldots + x_j).$$

Deduce a formula for the characteristic polynomial $\chi_{\mathcal{A}}$ of the braid arrangement. ✎

3.1. Combinatorial simpliciality.

Definition 3.10. Let $\mathcal{A}$ be a central essential arrangement in $\mathbb{R}^r$ and $0 \le n \le r-1$. Call

$$\mathcal{C}_n(\mathcal{A}) := \bigcup_{X\in L(\mathcal{A}),\ r(X)=r-n-1} \mathcal{K}(\mathcal{A}^X)$$

the set of **n-cells** of $\mathcal{A}$, and write

$$c_n := |\mathcal{C}_n(\mathcal{A})| = \sum_{X\in L(\mathcal{A}),\ r(X)=r-n-1} |\mathcal{K}(\mathcal{A}^X)|.$$

Notice that $\mathcal{C}_{r-1}(\mathcal{A})$ is the set of chambers. A **wall** of $\mathcal{A}$ is an $(r-2)$-cell of $\mathcal{A}$.

Proposition 3.11. Let $\mathcal{A}$ be a central essential arrangement of hyperplanes in $\mathbb{R}^r$, $r \ge 2$. Then $\mathcal{A}$ is simplicial if and only if $rc_{r-1} = 2c_{r-2}$.

Proof. Notice that

$$2c_{r-2} = \sum_{K\in\mathcal{K}(\mathcal{A})} |\{(r-2)\text{-cells adjacent to } K\}| \ge rc_{r-1}$$

and that the inequality is an equality if and only if $\mathcal{A}$ is simplicial. ☐

Corollary 3.12. Let $\mathcal{A}$ be a central essential arrangement of hyperplanes in $\mathbb{R}^r$. Then $\mathcal{A}$ is simplicial if and only if

$$\sum_{n=0}^{r-3}(-1)^n c_n = 1 + (-1)^{r-1} + (-1)^{r-1}\frac{r-2}{r}c_{r-2}.$$

In particular, if $r = 3$, then $\mathcal{A}$ is simplicial if and only if

$$3c_0 = 6 + c_1.$$

Proof. Since the Euler characteristic of the $(r-1)$-sphere is $1 + (-1)^{r-1}$, we have $\sum_n (-1)^n c_n = 1 + (-1)^{r-1}$. ☐

Corollary 3.13. Let $\mathcal{A}$ be a central essential arrangement of hyperplanes in $\mathbb{R}^r$, $r \ge 2$. Then $\mathcal{A}$ is simplicial if and only if

$$r\chi_{\mathcal{A}}(-1) + 2\sum_{H\in\mathcal{A}} \chi_{\mathcal{A}^H}(-1) = 0. \tag{6}$$

Proof. By Zaslavsky's Theorem (Thm. 3.8) the number of chambers of an arrangement $\mathcal{A}$ is $(-1)^r\chi_{\mathcal{A}}(-1)$. The number c_{r-2} is the number of walls of $\mathcal{A}$, i.e. the sum over the numbers of chambers for each restriction $\mathcal{A}^H$, $H \in \mathcal{A}$. ☐

Notice that Equation 6 does not depend on simplicial cones or on the fact that V is a vector space over the real numbers. This motivates the following definition.

Definition 3.14. Let K be a field and let $\mathcal{A}$ be a finite set of hyperplanes in $V = K^r$. We call $\mathcal{A}$ **(combinatorially) simplicial** if $\mathcal{A}$ is a central, essential arrangement satisfying $r\chi_{\mathcal{A}}(-1)+2\sum_{H\in\mathcal{A}}\chi_{\mathcal{A}^H}(-1) = 0$.

Remark 3.15. Since combinatorial simpliciality is equivalent to simpliciality for arrangements over $\mathbb{R}$, and since there is no definition of simpliciality for arrangements over other fields, we will call combinatorially simplicial arrangements simplicial for short.

Assume now that V is a vector space of dimension 3. By Corollary 3.12, we may rephrase simpliciality for the projective plane in the following way:

Corollary 3.16. Let $\mathcal{A}$ be a central essential arrangement in $V = K^3$. Let P denote the set of one-dimensional intersections of hyperplanes of $\mathcal{A}$. Then $\mathcal{A}$ is simplicial if and only if

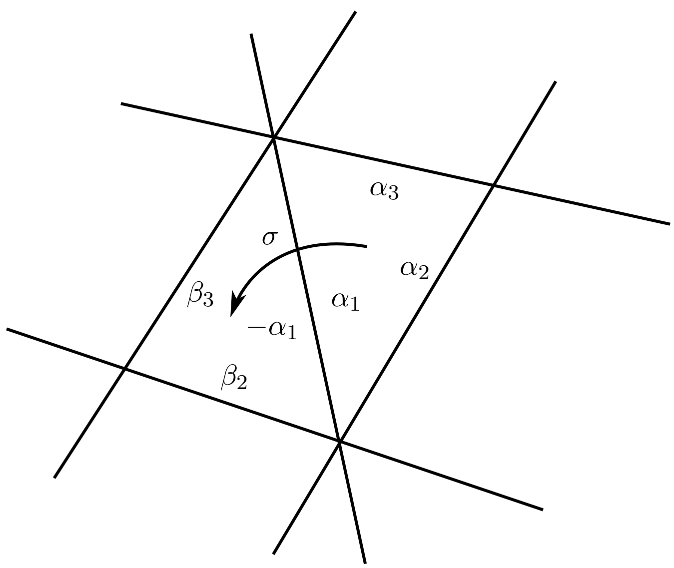
$$3(|P|-1) = \sum_{v\in P} |\{H \in \mathcal{A} \mid v \subset H\}|. \tag{7}$$

In other words, if $\mathcal{A}$ is simplicial, then up to an offset, in average the intersection points are triple points.

4. Simplicial arrangements and reflections

Example 4.1. Let W be a real reflection group acting on $V = \mathbb{R}^r$, i.e. a finite group generated by reflections on V. Let $\mathcal{R} \subseteq V^*$ be the set of roots of W. Then $\mathcal{A} = \{\ker\alpha \mid \alpha \in \mathcal{R}\}$ is a simplicial arrangement.
The reflection arrangement is the most symmetric type of simplicial arrangement, one cannot "distinguish" the chambers, they all look the same.

Definition 4.2. Let V be a finite dimensional vector space. A **reflection** is a non-trivial automorphism of V of finite order which fixes a hyperplane. If V has dimension ℓ, then the eigenspace of a reflection to the eigenvalue 1 is the fixed hyperplane (of dimension $\ell-1$), and there is a one dimensional eigenspace to an eigenvalue ζ which is a root of unity.



Lemma 4.3. Let $\mathcal{A}$ be a simplicial arrangement and K a chamber, i.e. there is a basis $B^\vee = \{\alpha_1^\vee,\dots,\alpha_r^\vee\}$ of V such that $K = \langle B^\vee\rangle_{>0}$. Let $\tilde{K}$ be the chamber with

$$\overline{K} \cap \overline{\tilde{K}} = \langle \alpha_2^\vee,\dots,\alpha_r^\vee\rangle_{\geq 0}.$$

Then there is a unique $\beta^\vee \in V$ with

$$\tilde{K} = \langle \tilde{B}^\vee\rangle_{>0}, \quad \tilde{B}^\vee = \{\beta^\vee,\alpha_2^\vee,\dots,\alpha_r^\vee\}, \quad \text{and} \quad |B\cap -\tilde{B}| = 1,$$

where $B := (B^\vee)^*$ and $\tilde{B} := (\tilde{B}^\vee)^*$ denote the dual bases.

Proof. Choose $\beta^\vee \in V$ such that $\tilde{K} = \langle \beta^\vee,\alpha_2^\vee,\dots,\alpha_r^\vee\rangle_{>0}$. Let $\mu_1,\dots,\mu_r \in \mathbb{R}$ be such that $\beta^\vee = \sum_{i=1}^r \mu_i\alpha_i^\vee$ (notice $\mu_1 \neq 0$). Let $\tilde{B} = \{\beta_1,\dots,\beta_r\}$ be the dual basis of $\{\beta^\vee,\alpha_2^\vee,\dots,\alpha_r^\vee\}$, and $B = \{\alpha_1,\dots,\alpha_r\}$ be dual to $B^\vee$. Then $\beta_1 = \frac{1}{\mu_1}\alpha_1$ and $\beta_j = -\frac{\mu_j}{\mu_1}\alpha_1 + \alpha_j$ for $j>1$. To obtain $|B\cap -\tilde{B}| = 1$ we need $-\alpha_1 = \beta_1 \in \tilde{B}$ and hence $\mu_1 = -1$, $\beta_1 = -\alpha_1$ and $\beta_j = \mu_j\alpha_1 + \alpha_j$ for $j>1$. Thus a $\beta^\vee$ as desired exists and is unique. ∎

Corollary 4.4. Using the notation of the proof of the Lemma, the map

$$\sigma : V^* \to V^*, \quad \alpha_i \mapsto \beta_i$$

is a reflection. With respect to $B = (B^\vee)^*$, it becomes the matrix

$$\begin{pmatrix} -1 & \mu_2 & \dots & \mu_r \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

Example 4.5. Let $R = \{(1,0),(0,1),(1,2)\} \in (\mathbb{R}^2)^*$, $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$.

Then $K = \langle B^\vee \rangle_{>0}$ is a chamber if $B^\vee = \{\alpha_1^\vee = (1,0), \alpha_2^\vee = (0,1)\}$, $K' = \langle \tilde{B}^\vee \rangle_{>0}$ with $\tilde{B}^\vee = \{\tilde{\beta}^\vee = (-2,1), \alpha_2^\vee = (0,1)\}$ is an adjacent chamber.

To obtain $\mu_1 = -1$, we need to choose $\beta^\vee = (-1, \frac{1}{2})$, hence $\mu_2 = \frac{1}{2}$. The unique reflection σ is

$$\begin{pmatrix} -1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

with respect to $B = (B^\vee)^*$.

4.1. Reflections and Cartan matrices.

Definition 4.6. Let $\mathcal{A}$ be a simplicial arrangement, $K = \langle B^\vee \rangle_{>0}$, $B^\vee = \{\alpha_1^\vee, \dots, \alpha_r^\vee\}$ a chamber, and $B = \{\alpha_1, \dots, \alpha_r\}$ be dual to $B^\vee$.

By Corollary 4.4: for K, B there are unique reflections $\sigma_1, \dots, \sigma_r$, represented by

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ \mu_{i,1} & \cdots & -1 & \cdots & \mu_{i,r} \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix},$$

for certain $\mu_{i,j} \in \mathbb{R}$, $i \neq j$ with respect to B.

The matrix $C^{K,B} = (c_{i,j})_{1 \le i,j \le r}$ with

$$c_{i,j} := \begin{cases} -\mu_{i,j} & \text{if } i \neq j \\ 2 & \text{if } i = j \end{cases}$$

is called the **Cartan matrix** of (K,B) in $\mathcal{A}$. Note that

$$\sigma_i(\alpha_j) = \alpha_j - c_{i,j}\alpha_i$$

for all $1 \le i,j \le r$.

We sometimes write $\sigma_i^{K,B}$ to emphasize that σ_i depends on K and B.

Example 4.7. (1) Let $\mathcal{A}$ be as in the last example. Then the Cartan matrix of (K,B) is

$$C^{K,B} = \begin{pmatrix} 2 & -\frac{1}{2} \\ -2 & 2 \end{pmatrix}.$$

(2) If W is a Weyl group with root system $\mathcal{R}$, then all Cartan matrices of (K,B) when B is a set of simple roots for the chamber K are equal and coincide with the classical Cartan matrix of W.

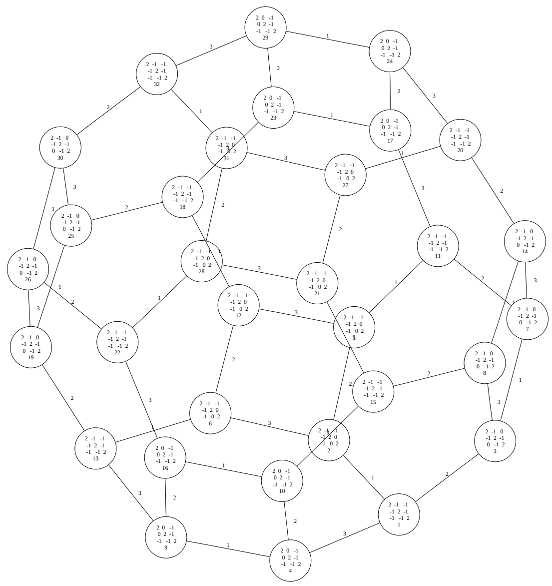


FIGURE 4. A Cartan graph

4.2. Categories.

Definition 4.8. A **category** $\mathcal{C}$ consists of a class (*not a set!*) of **objects** $\mathrm{Obj}\,\mathcal{C} = \{A, B, C, \ldots\}$ and of sets $\mathrm{Mor}(A, B)$ for each pair (A, B) of objects, such that:

(1) $\mathrm{Mor}(A, B) \cap \mathrm{Mor}(C, D) = \varnothing$ if $(A, B) \neq (C, D)$.
(2) For each triple (A, B, C) of objects, there is a map

$$\circ : \mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C), \quad (g, f) \mapsto g \circ f$$

satisfying
  (a) $h \circ (g \circ f) = (h \circ g) \circ f$ (**associativity**)
  (b) for each $A \in \mathrm{Obj}\,\mathcal{C}$ there is a unique element $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ with

$$\begin{aligned} f \circ \mathrm{id}_A &= f \quad \text{for all} \quad f \in \mathrm{Mor}(A, B), \\ \mathrm{id}_A \circ g &= g \quad \text{for all} \quad g \in \mathrm{Mor}(B, A). \end{aligned}$$

The elements of $\mathrm{Mor}(A, B)$ are called **morphisms** from A to B.

Example 4.9. (1) **category of sets**: objects are the sets, morphisms are maps.
(2) **category of groups**: objects are groups, morphisms are homomorphisms of groups.
(3) K a field, **category of vector spaces** over K: objects are K-vector spaces, morphisms are linear maps.

Definition 4.10. Let $A, B \in \mathrm{Obj}\,\mathcal{C}$ be objects in a category $\mathcal{C}$. A morphism $f \in \mathrm{Mor}(A, B)$ is called an **isomorphism** if and only if there exists a $g \in \mathrm{Mor}(B, A)$ such that

$$g \circ f = \mathrm{id}_A, \quad f \circ g = \mathrm{id}_B.$$

An isomorphism $f \in \mathrm{Mor}(A, A)$ is called an **automorphism**. We write

$$\mathrm{Aut}(A) := \{f \mid f \text{ automorphism of } A\};$$

this is a group with respect to $\circ$.

4.3. Reflection groupoids.

Definition 4.11.

(1) A category with only one object in which every morphism is an isomorphism is a **group**.

(2) A category with only one object is a **monoid**.
(3) A category in which every morphism is an isomorphism is a **groupoid**.

Example 4.12. (1) If X is a topological space, then the set of paths in X up to homotopies form a groupoid with concatenation: the objects are the points of X, morphisms are the homotopy classes of paths.
(2) If X is a topological space and $x \in X$, then closed paths at x up to homotopies form a group with concatenation.

Example 4.13. Let $(G, \circ)$ be a group in the classical sense.

(1) The category $\mathcal{C}$ with $\mathrm{Obj}(\mathcal{C}) = \{A\}$ and $\mathrm{Mor}(A, A) = G$ is a group. Notice that $\mathrm{Aut}(A) = G$.
(2) The category $\mathcal{C}$ with $\mathrm{Obj}(\mathcal{C}) = G$ and $\mathrm{Mor}(g, h) = \{h \circ g^{-1}\}$ for $g, h \in G$ is a groupoid. Here, $\mathrm{Aut}(g) = \{\mathrm{id}\}$ for all $g \in G$.
For example, $G = (\mathbb{Z}/3\mathbb{Z}, +)$...

Definition 4.14. A groupoid $\mathcal{C}$ is called **connected** if for any two objects A, B there exists a morphism in $\mathrm{Mor}(A, B)$.
Two groupoids $\mathcal{C}_1$ and $\mathcal{C}_2$ are called **isomorphic** if there exists an invertible **functor** $F : \mathcal{C}_1 \to \mathcal{C}_2$, i.e. F associates to each $A \in \mathcal{C}_1$ an object $F(A)$ in $\mathcal{C}_2$, and to each morphism $f \in \mathrm{Mor}(A, B)$ a morphism $F(f) \in \mathrm{Mor}(F(A), F(B))$ in such a way that

(1) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathrm{Obj}(\mathcal{C}_1)$,
(2) $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, $A, B, C \in \mathrm{Obj}(\mathcal{C}_1)$.

We then write $\mathcal{C}_1 \cong \mathcal{C}_2$.

Proposition 4.15. Let $\mathcal{C}$ be a connected groupoid. Then $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$ for any $A, B \in \mathrm{Obj}(\mathcal{C})$.

Proof. Let $A, B \in \mathrm{Obj}(\mathcal{C})$. Since $\mathcal{C}$ is connected, there exists a morphism $f \in \mathrm{Mor}(A, B)$ which is an isomorphism because $\mathcal{C}$ is a groupoid. But then the map

$$\mathrm{Aut}(A) \to \mathrm{Aut}(B), \quad g \mapsto fgf^{-1}$$

is an isomorphism. □

Definition 4.16. Let $\mathcal{A}$ be a simplicial arrangement in $V = \mathbb{R}^r$. We construct a category $\mathcal{C}(\mathcal{A})$ with

- objects: $\mathrm{Obj}(\mathcal{C}(\mathcal{A})) = \{B = (\alpha_1, \dots, \alpha_r) \in (V^*)^r \mid \langle B^* \rangle_{>0} \in \mathcal{K}(\mathcal{A})\}$ (where the bases B are ordered).
- morphisms: for each $B = (\alpha_1, \dots, \alpha_r) \in \mathrm{Obj}(\mathcal{C}(\mathcal{A}))$ and $i = 1, \dots, r$ there is a morphism $\sigma_i^{K,B} \in \mathrm{Mor}(B, (\sigma_i^{K,B}(\alpha_1), \dots, \sigma_i^{K,B}(\alpha_r)))$.
All other morphisms are compositions of the generators $\sigma_i^{K,B}$.

A **reflection groupoid** $\mathcal{W}(\mathcal{A})$ of $\mathcal{A}$ is a connected component of $\mathcal{C}(\mathcal{A})$.
A **Weyl groupoid** [1] is a reflection groupoid for which all Cartan matrices are integral.

Using the so-called gate property, one can prove the existence of a type function for the chamber complex of a simplicial arrangement. In other words:

Proposition 4.17. Let $\mathcal{A}$ be a simplicial arrangement, $\mathcal{W}(\mathcal{A})$ a reflection groupoid, and $B_1 = (\alpha_1, \dots, \alpha_r)$, $B_2 = (\beta_1, \dots, \beta_r)$ two objects with $\langle B_1^* \rangle_{>0} = \langle B_2^* \rangle_{>0}$. Then there exist $\lambda_1, \dots, \lambda_r$ such that $\alpha_i = \lambda_i \beta_i$ for all $i = 1, \dots, r$.
In particular, for a fixed reflection groupoid we obtain a unique labelling of the walls of each chamber with the labels $1, \dots, r$.

[1] This is not the general definition of a Weyl groupoid. For a complete set of axioms, see [?].

Definition 4.18. Let $\mathcal{A}$ be a simplicial arrangement, $\mathcal{W}(\mathcal{A})$ a reflection groupoid, and $K = \langle B^* \rangle_{>0}$ a chamber for $B = (\alpha_1, \dots, \alpha_r) \in \mathrm{Obj}(\mathcal{W}(\mathcal{A}))$.
For $i \in \{1, \dots, r\}$, let $\rho_i(K)$ be the chamber adjacent to K with common wall $\ker \alpha_i$. We thus obtain well defined maps

$$\rho_i : \mathcal{K}(\mathcal{A}) \mapsto \mathcal{K}(\mathcal{A})$$

which satisfy $\rho_i^2 = \mathrm{id}$ by the proposition.

5. Crystallographic arrangements

Definition 5.1 ([?])**.** Let $\mathcal{A}$ be a simplicial arrangement in V and $\mathcal{R} \subseteq V^*$ a finite set such that $\mathcal{A} = \{\ker \alpha \mid \alpha \in \mathcal{R}\}$ and $\mathbb{R}\alpha \cap \mathcal{R} = \{\pm\alpha\}$ for all $\alpha \in \mathcal{R}$.

We call $(\mathcal{A}, V, \mathcal{R})$ a **crystallographic arrangement** if for all chambers $K \in \mathcal{K}(\mathcal{A})$:

$$\mathcal{R} \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha, \tag{8}$$

where

$$B^K = \{\alpha \in \mathcal{R} \mid \forall x \in K \ : \ \alpha(x) \geq 0, \ \langle \ker \alpha \cap \overline{K} \rangle = \ker \alpha\}$$

corresponds to the set of walls of K.

Definition 5.2. Two crystallographic arrangements $(\mathcal{A}, V, \mathcal{R})$, $(\mathcal{A}', V, \mathcal{R}')$ in V are called **equivalent** if there exists $\psi \in \mathrm{Aut}(V^*)$ with $\psi(\mathcal{R}) = \mathcal{R}'$. We then write $(\mathcal{A}, V, \mathcal{R}) \cong (\mathcal{A}', V, \mathcal{R}')$.

If $\mathcal{A}$ is an arrangement in V for which a set $\mathcal{R} \subseteq V^*$ exists such that $(\mathcal{A}, V, \mathcal{R})$ is crystallographic, then we say that $\mathcal{A}$ is **crystallographic**.

Example 5.3.
1. Let $\mathcal{R}$ be the set of roots of the root system of a crystallographic reflection group (i.e. a Weyl group). Then $(\{\ker \alpha \mid \alpha \in \mathcal{R}\}, V, \mathcal{R})$ is a crystallographic arrangement.
2. If $R_+ := \{(1,0), (3,1), (2,1), (5,3), (3,2), (1,1), (0,1)\}$, then $(\{\alpha^\perp \mid \alpha \in R_+\}, \mathbb{R}^2, R_+ \cup -R_+)$ is a crystallographic arrangement.

Definition 5.4. Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement and K a chamber. Fixing an ordering for B^K, we obtain a unique reflection groupoid $\mathcal{W}(\mathcal{A})$ and thus unique orderings for all $B^{K'}$, $K' \in \mathcal{K}(\mathcal{A})$ (type function). Hence we obtain a unique coordinate map

$$\Upsilon^K : V \to \mathbb{R}^r \quad \text{with respect to} \quad B^K.$$

The elements of the standard basis $\{\alpha_1, \dots, \alpha_r\} = \Upsilon^K(B^K)$ are called **simple roots**. The set

$$R^K := \{\Upsilon^K(\alpha) \mid \alpha \in \mathcal{R}\} \subseteq \mathbb{N}_0^r \cup -\mathbb{N}_0^r$$

is called the set of **roots** of $\mathcal{A}$ at K. The roots in $R_+^K := R^K \cap \mathbb{N}_0^r$ are called **positive**.

Let $1 \leq i, j \leq r$. Then it is easy to see that

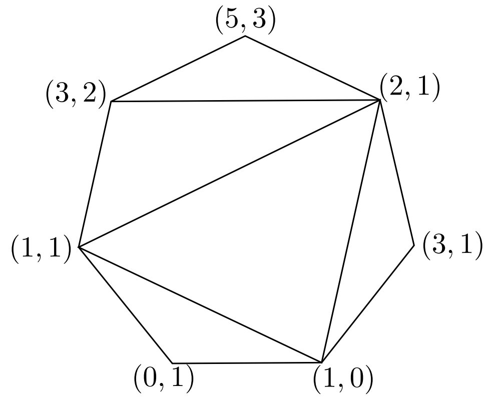
$$c_{i,j}^K = \begin{cases} -\max\{k \in \mathbb{N}_{\geq 0} \mid k\alpha_i + \alpha_j \in R^K\} & i \neq j \\ 2 & i = j \end{cases},$$

where $C^K := (c_{i,j}^K)_{i,j}$ is the Cartan matrix of (K, B^K).

Theorem 5.5. Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement, K a chamber, and $\alpha \in R_+^K$ a positive root. Then either α is simple, or it is the sum of two positive roots in R_+^K.

Example 5.6.

$$R_+ := \{(1,0), (3,1), (2,1), (5,3), (3,2), (1,1), (0,1)\}$$



Using determinants and volumes one can prove:

Theorem 5.7 (Cuntz, Heckenberger (2015); Cuntz (2019))**.** Let $r > 2$. Then there are only finitely many equivalence classes of irreducible crystallographic arrangements of rank r.

An enumeration with a computer gives:

Theorem 5.8 (Cuntz, Heckenberger, 2009/2010)**.** There are exactly three families of crystallographic arrangements:

(1) The family of rank two parametrized by triangulations of a convex n-gon by non-intersecting diagonals.
(2) For each rank $r > 2$, arrangements of type A_r, B_r, C_r and D_r, and a further series of $r-1$ arrangements.
(3) Further 74 "sporadic" arrangements of rank r, $3 \le r \le 8$.

6. Generalized root systems

Definition 6.1 (Dimitrov, Fioresi [?])**.** Let $(V, (\cdot,\cdot))$ be a finite dimensional euclidean vector space, $\varnothing \neq R \subseteq V$ a finite subset. The pair (R, V) is called a **generalized root system** (GRS) if $V = \langle R \rangle$ and for all $\alpha, \beta \in R$:

$$\begin{aligned}(\alpha,\beta) < 0 &\implies \alpha+\beta \in R,\\ (\alpha,\beta) > 0 &\implies \alpha-\beta \in R,\\ (\alpha,\beta) = 0 &\implies (\alpha+\beta \in R \Longleftrightarrow \alpha-\beta \in R).\end{aligned}$$

$\alpha \in R$ is called a **root**, the **rank** of (R, V) is the dimension of V.

Lemma 6.2. Let (R, V) be a GRS. Then the following hold.

(1) $R = -R$.
(2) $\forall 0 \neq \alpha \in R\ \exists \beta \in R,\ k \in \mathbb{N}$: $\quad \mathbb{R}\alpha \cap R = \{j\beta \mid j \in \mathbb{Z}, -k \le j \le k\}$.
β is called **primitive**, k is the **multiplier** of β.

Example 6.3. (i) If $\mathcal{A}$ is a Weyl arrangement with root system R, then $R \cup \{0\}$ is a GRS.
(ii) Let $B \subseteq \Delta$ be a subset of a simple system, $X := \langle B \rangle \le V$, and $\pi : V \to V/X$ the projection. Then $\pi(R \cup \{0\})$ is a GRS. The corresponding arrangement is the restriction $\mathcal{A}^{X^\perp}$. Dimitrov and Fioresi call $\pi(R \cup \{0\})$ a **quotient of a root system**.

Theorem 6.4 (Cuntz, Mühlherr [?])**.** Each irreducible GRS of rank at least 2 is equivalent to a quotient of a classic root system of a finite Weyl group.

CHAPTER 3

Matroids

From now on we assume that all arrangements are **central**.

1. Geometric lattices

Recall the reflection arrangement $\mathcal{A}$ of type A_3 in $\mathbb{R}^4$ given by the defining equation

$$Q(\mathcal{A}) : (x-y)(x-z)(x-w)(y-z)(y-w)(z-w).$$

A projectivized picture is given below:

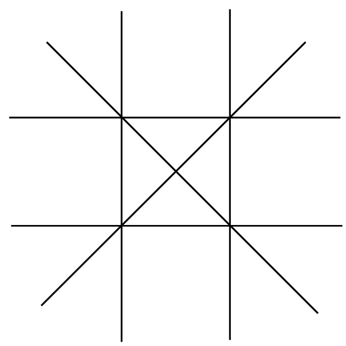


FIGURE 1. The arrangement of type A_3.

The arrangement has the intersection lattice $L(\mathcal{A}_3)$:

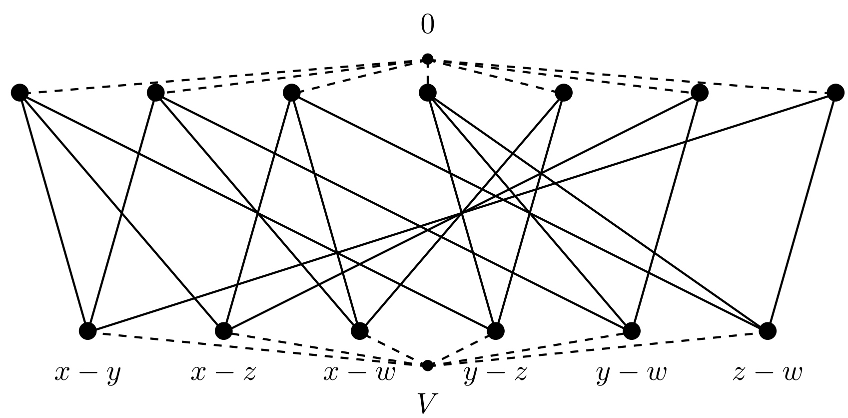


FIGURE 2. The intersection lattice $L(\mathcal{A})$ of the reflection arrangement of type A_3.

The goal of this lecture is to understand the special properties of this poset. Which posets can occur as the intersection lattice of some arrangement? This leads to the more abstract notion of **matroids** which is class of posets satisfying a set of core properties of intersection lattices.

In a poset P, the element X **covers** Y, written as $X :> Y$, if $X > Y$ and there is no other element Z distinct from X and Y with $X > Z > Y$.

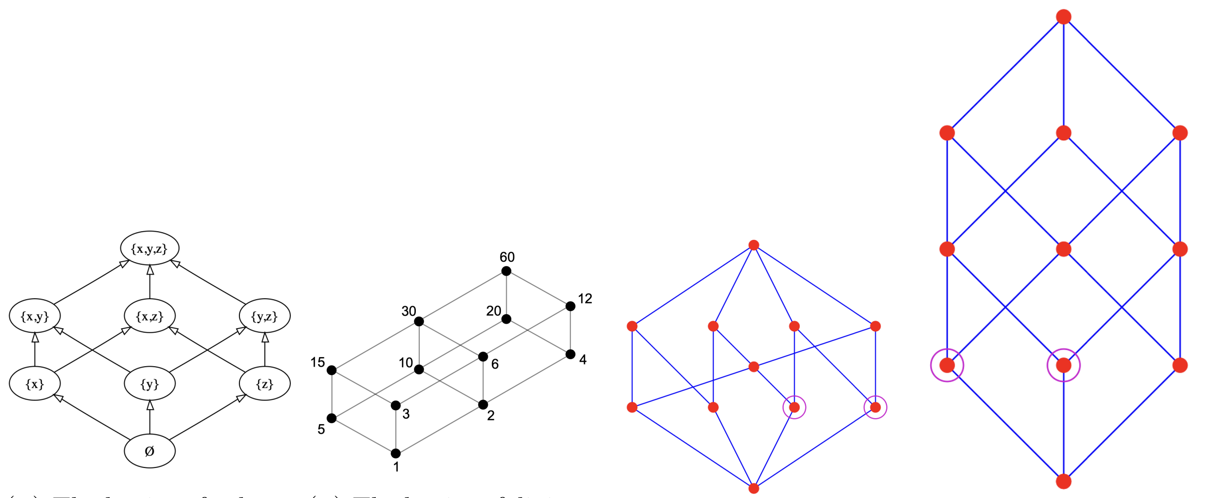
We know that for $X \in L(\mathcal{A})$, the **rank** $r(X)$ of X is defined as $r(X) := \ell - \dim X$, i.e. the codimension of X. This turns $L(\mathcal{A})$ into a **graded poset**, i.e., a poset with a rank function r compatible with the order and for a cover relation $X :> Y$ it holds that $r(X) = r(Y) + 1$.

For two elements $X, Y \in P$ in a poset P, Z is a least upper bound, called a **join** or supremum and denoted by $X \vee Y$ if $X < Z$, $Y < Z$ for every Z' with this property we have $Z < Z'$. Analogously, a greatest lower bound of $X, Y \in P$ is called **meet** or infimum and dented by $X \wedge Y$.

Proposition 1.1. *Let $X, Y \in L(\mathcal{A})$ be two elements in an intersection lattice. The pair X, Y has both a join $X \vee Y$ and a meet $X \wedge Y$.*

PROOF. Set $X \vee Y$ to be $X \cap Y$ and $X \wedge Y$ to be $X + Y$. □

A poset in which two elements have both a join and a meet is called a **lattice**. A finite lattice has a unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$. The **atoms** are the elements that cover $\hat{0}$. For a finite graded lattice L we assume throughout that $r(\hat{0}) = 0$. Hence the atoms are the elements of rank 1.



(A) The lattice of subsets of $\{x, y, z\}$.
(B) The lattice of divisors of 60.
(C) Not a lattice.
(D) Not a lattice.

Proposition 1.2. *Let L be a finite graded lattice. The following two conditions are equivalent:*

(1) For all $X, Y \in L$, we have $r(X) + r(Y) \geq r(X \wedge Y) + r(X \vee Y)$.
(2) If X and Y both cover $X \wedge Y$ then $X \vee Y$ covers both X and Y.

PROOF. Assume (1). Suppose $X, Y :> X \wedge Y$ and thus

$$r(X) = r(Y) = r(X \wedge Y) + 1 \quad \text{and} \quad r(X \vee Y) > r(X) = r(Y).$$

By (1) we get

$$\begin{aligned} &r(X) + r(Y) \geq r(X) - 1 + r(X \vee Y). \\ \Rightarrow &r(Y) \geq r(X \vee Y) - 1. \\ \Rightarrow &X \vee Y :> Y. \end{aligned}$$

For (2) implies (1) see Stanley, Enumerative Combinatorics, Prop. 3.3.2. □

Definition 1.3. A finite lattice L satisfying (1) or (2) above is called **(upper) semimodular**. A finite lattice L is **atomic** if every element is the join of some atoms. A finite lattice L is **geometric** if it is both semimodular and atomic.

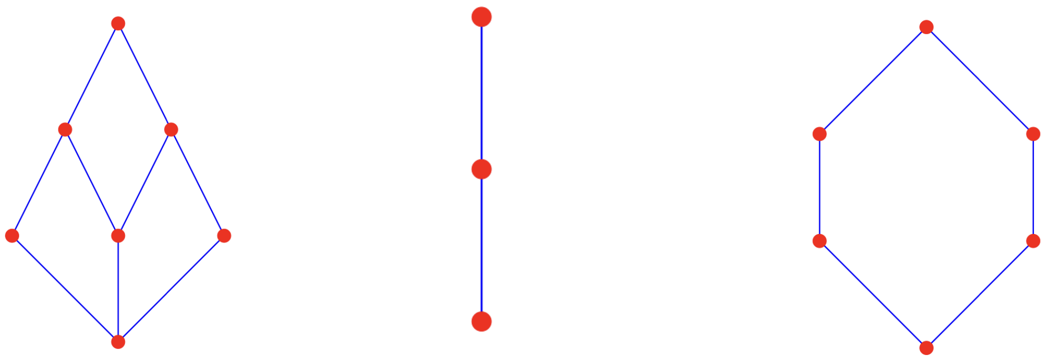


FIGURE 4. Three nongeometric lattices.

Example 1.1. The following lattices are **not** geometric lattices. The first one is not semimodular, the second one is not atomic and the third one is not atomic and not semimodular.

Proposition 1.4. *The intersection lattice of an arrangement is a geometric lattice.*

PROOF. As every element in an intersection lattice is the intersection of some hyperplanes, i.e., atoms in the lattice, it is atomic. The dimension formula for subspaces from linear algebra implies that it is semimodular. □

There are several cryptomorphic[1] ways of defining a matroid. Geometric lattices are on way of defining (simple) matroids. We will see more equivalent ways of defining a matroid soon.

Proposition 1.5. *Let E be the set of atoms of a geometric lattice L. Define a function $f : \mathcal{P}(E) \to \mathbb{Z}_{\geq 0}$ by setting for $A \subseteq E$*

$$f(A) := r\left(\bigvee_{a \in A} a\right).$$

This function f satisfies

(R1) $0 \leq f(A) \leq |A|$,
(R2) $f(A) \leq f(B)$ *whenever* $A \subseteq B$ *and* (*f is **monotone***)
(R3) $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$. (*$f$ is **submodular***)

PROOF. $0 \leq f(A)$ and $f(A) \leq f(B)$ for $A \subseteq B$ is clear.

Suppose $A = \{a_1, \ldots, a_k\}$. As L is semimodular we have

$$\begin{aligned} f(\{a_1, \ldots, a_{k-1}\}) + f(\{a_k\}) &\geq f(\{a_1, \ldots, a_k\}) + f(\varnothing) \\ \Rightarrow f(\{a_1, \ldots, a_{k-1}\}) + 1 &\geq f(A). \end{aligned}$$

Iterating this process yields $|A| \geq f(A)$.

Let $A, B \subseteq E$ be arbitrary subsets. Using again that L is semimodular yields

$$\begin{aligned} f(A) + f(B) &= r(\bigvee_{a \in A} a) + r(\bigvee_{b \in B} b) \\ &\geq r(\bigvee_{a \in A} a \vee \bigvee_{b \in B} b) + r(\bigvee_{a \in A} a \wedge \bigvee_{b \in B} b) \end{aligned}$$

[1] secretly or non-trivially equivalent

The first term in the last line is $f(A \cup B)$. As we have $\bigvee_{a\in A} a \wedge \bigvee_{b\in B} b > \bigvee_{c\in A\cap B} c$ this in total implies (R3). $\square$

Definition 1.6. A **matroid** is a pair $M = (E, r)$ of a finite ground set E together with a rank function $r : \mathcal{P}(E) \to \mathbb{Z}_{\geq 0}$ that satisfies the three axioms (R1), (R2) and (R3).

We can recover the lattice underlying a matroid as follows:

THEOREM 1.7. *Let $M = (E, r)$ be a matroid. A subset $F \subseteq E$ is called a* ***flat*** *if $r(F) < r(F \cup e)$ for all $e \notin F$. We define $L(M)$ to be the poset of flats of M ordered by inclusion.*

The poset $L(M)$ is a geometric lattice.

A matroid $M = (E, r)$ is called **simple** if $r(A) = |A|$ for all $A \subseteq E$ with $|A| \leq 2$. We leave it as an easy exercise to check that all matroids arising from Proposition 1.5 are simple. This yields:

Corollary 1.8. *There is a one-to-one correspondence between geometric lattices and simple matroids.*

A main source of examples of matroids comes from arrangements:

Example 1.2. Let $\mathcal{A} = \{H_e\}e \in E$ for some finite set E be an arrangement in the vector space K^ℓ.

This yields a matroid $M(\mathcal{A}) = (E, r)$ defined via the usual rank function

$$r(A) := \ell - \dim \left(\bigcap_{e\in A} H_e \right).$$

The above discussion confirms that $M(\mathcal{A})$ is indeed a matroid, i.e., r satisfies (R1),(R2),(R3).

Matroids arising in this way are called **representable over** K. For instance the matroid associated to the geometric lattice in Figure 2 is representable over $\mathbb{R}$ via the braid arrangement.

Remark on posets and $<$ vs $\leq$.

2. More on matroids: Independence

Recall the definition of a a matroid via its rank function. Today we cover alternative but equivalent ways of defining matroids via their independence properties.

Example 2.1. To start consider these three examples:

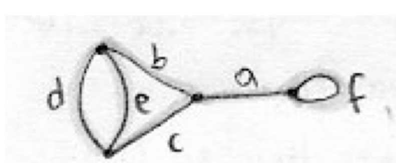
Linear algebra: Consider the vectors

$$a = (1, 00), b = (0, 1, 0), c = (0, 0, 1), d = (0, \frac{1}{2}, \frac{1}{2}), e = (0, 1, 1), f = (0, 0, 0).$$

Goal: Choose a set of linear independent vectors.

$$\begin{aligned}&\varnothing\\ &a, b, c, d, e\\ &ab, ac, ad, ae, bc, bd, be, cd, ce\\ &abc, abd, abe, acd, ace.\end{aligned}$$

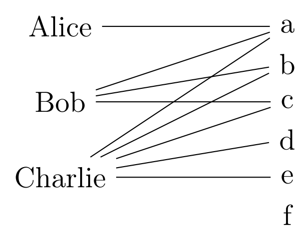
Graph theory: Consider the graph



Goal: Choose a set of edges without cycles.

This yields the same set of edges as above.

Matching theory: Consider the bipartite graph of Alice, Bob and Charlie with their preferred movies $a, \ldots, f$.



Goal: Choose a set of movies, so that the set of chosen movies can be assigned to Alice, Bob and Charlie taking their preferences into account.

This yields the same set of movies as above.

These are three instances of independence captured by matroids:

Definition 2.1. A **matroid** $M = (E, \mathcal{I})$ is a pair where

- E is a finite set and
- $\mathcal{I}$ is a collection of subsets of E such that
  (I1) $\varnothing \in \mathcal{I}$,
  (I2) If $I \subseteq J$ and $J \in \mathcal{I}$ then $I \in \mathcal{I}$ and
  (I3) If $I, J \in \mathcal{I}$ and $|I| < |J|$ then there exists a $j \in J \setminus I$ such that $I \cup j \in \mathcal{I}$.

We call E the **ground set** and $\mathcal{I}$ the collection of **independent sets**.

Example 2.2. Let $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$. Let E be the set $\{1, \ldots, n\}$ and $\mathcal{I}$ all subsets of E of size at most m. Then $U_{m,n} = (E, \mathcal{I})$ is a matroid, called the **uniform matroid**.

Proposition 2.2. *Let E be a finite set of vectors in a vector space V and let $\mathcal{I}$ be the collection of linear independent sets in E. Then $(E, \mathcal{I})$ is a matroid.*

Proof. (I1) and (I2) are clear. So consider $I, J \in \mathcal{I}$ with $|I| < |J|$. Assume that $I \cup j$ is linear dependent for all $j \in J \setminus I$. This implies that

$$\operatorname{Span}(J) \subseteq \operatorname{Span}(I).$$
$$\Rightarrow |J| = \dim(\operatorname{Span}(J)) \leq \dim(\operatorname{Span}(I)) = |I|$$

which is a contradiction to our assumption. □

Proposition 2.3. *Let $G = (V, E)$ be an undirected graph and let $\mathcal{I}$ be the collection of edges without cycles. Then $(E, \mathcal{I})$ is a matroid.*

Proof. (I1) and (I2) is trivial. For (I3) we start with the observation that if $I \subseteq E$ is independent, then the graph (V, I) has $|V| - |I|$ connected components.

So consider again $I, J \in \mathcal{I}$ with $|I| < |J|$ and assume that $I \cup j \notin \mathcal{I}$ for all $j \in J \setminus I$. Hence $I \cup j$ contains a cycle for all $j \in J \setminus I$. Thus adding j doesn't connect components of (V, I). So $(V, I \cup J)$ has $|V| - |I|$ components. Hence (V, J) has at least $|V| - |I|$ components. But by the above observation (V, J) has exactly $|V| - |J|$ components. So

$$|V| - |J| \geq |V| - |I|$$

which contradicts our assumption $|I| < |J|$. □

Recall that a **bipartite graph** G is a graph in which the vertices can be partitioned into two subsets (D, E) so that every edge in G connects a vertex in D with a vertex in E. A **matching** in a graph G is a collection of disjoint edges.

Proposition 2.4. *Let G be a bipartite graph with bipartition (D, E). Let $\mathcal{I}$ be the collection of subsets of E that can be matched to D. Then $(E, \mathcal{I})$ is a matroid.*

PROOF. Exercise. □

Definition 2.5.

- **Linear matroids** or **representable matroids** are matroids of vector configurations.
- **Graphical matroids** are matroids of graphs.
- **Transversal matroids** are matroids of matching problems.

Inspired by linear algebra we give a special name to the maximal independent sets:

Definition 2.6. A **basis** of a matroid is a maximal independent set.

Proposition 2.7. *All bases of a matroid have the same size. We call the size of a basis the* **rank** *of the matroid.*

PROOF. Follows immediately from (I3). □

For a linear matroid, a basis is a basis of the vector space.

For the graphic matroid of a connected graph, a basis is a **spanning tree**. Otherwise it is a **spanning forest**.

We could have also defined a matroid via its set of bases:

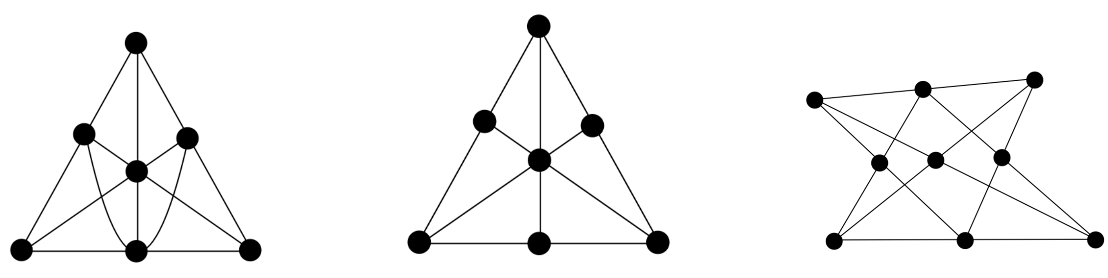
THEOREM 2.8. *Let E be a finite set and $\mathcal{B} \subseteq \mathcal{P}(E)$ a collection of subsets. Then $\mathcal{B}$ is collection of bases of a matroid if and only if*

(B1) $\mathcal{B} \neq \varnothing$ and
(B2) If $A, B \in \mathcal{B}$ and $a \in A \setminus B$ then there exists $b \in B \setminus A$ such that $A \setminus a \cup b \in \mathcal{B}$.

PROOF. Let $\mathcal{B}$ be the collection of bases of a matroid $(E, \mathcal{I})$. Property (I1) implies (B1). (B2) follows from (I3) applied to the sets $A \setminus a$ and B.

The converse is somewhat lengthy but not difficult. Can be looked up in Oxley's book. □

Example 2.3. We can "draw" rank 3 matroids as point-line-incidences where the points are E and the bases are the triples not that are contained in a line. Figure 5 shows three prominent matroids of rank 3 of size 7, 7 and 9.



(A) The Fano matroid. (B) The non-Fano matroid. (C) The non-Pappus matroid.

FIGURE 5. Three prominent matroids of rank 3.

Proposition 2.9. *(i) The Fano matroid is representable over a field $\mathbb{F}$ if and only if* $\mathrm{char}(\mathbb{F}) = 2$.

(ii) The non-Fano matroid is representable over a field $\mathbb{F}$ if and only if $\operatorname{char}(\mathbb{F}) \neq 2$.
(iii) The non-Pappus is not representable over any field.

Proof. (i) and (ii) are exercises. For (iii) note that a representation of the non-Pappus matroid over a field $\mathbb{F}$ would contradict the Pappus theorem which exactly asserts that in a drawing of nine points in the projective plane $\mathbb{P}^2(\mathbb{F})$ the three middle points must be collinear, i.e., don't form a basis. □

Theorem 2.10 (Nelson '18). *Asymptotically, almost all matroids are not representable over any field.*

This means that if M_n is the set of all matroids on the set $\{1, \ldots, n\}$ and R_n is the set of all matroids on $\{1, \ldots, n\}$ that are representable over some field, we have

$$\lim_{n\to\infty} \frac{|R_n|}{|M_n|} = 0.$$

Let us now finally return to the rank function of a matroid.

Definition 2.11. Let $M = (E, \mathcal{I})$ be a matroid. We define a rank function $r_M : \mathcal{P}(E) \to \mathbb{Z}_{\geq 0}$ by setting for $A \subseteq E$

$$r_M(A) := \max\{|I| : I \in \mathcal{I}, I \subseteq A\}.$$

Proposition 2.12. *The rank function r_M satisfies the axioms (R1), (R2) and (R3).*

Proof. (R1) and (R2) are clear. For (R3) consider $A, B \subseteq E$ and let $I \subseteq A \cap B$ be an inclusion maximal independent set so that $r_M(A \cap B) = |I|$. Using (I3) we can extend I to an inclusion maximal independent set $J \subseteq A \cup B$. Now we compute

$$r_M(A \cup B) = |J| = |J \cap A| + |J \cap B| - |J \cap (A \cap B)| \leq r_M(A) + r_M(B) - r_M(A \cap B).$$ □

We can recover the matroid $M = (E, \mathcal{I})$ from a rank function r as the independent sets are exactly those $I \subseteq E$ with $r(i) = |I|$.

Theorem 2.13. *Let E be a finite set. A function $r : \mathcal{P}(E) \to \mathbb{Z}_{\geq 0}$ is the rank function of a matroid if and only if it satisfies (R1), (R2) and (R3).*

We need the following lemma for the proof.

Lemma 2.14. *Let $r : \mathcal{P}(E) \to \mathbb{Z}_{\geq 0}$ be a function that satisfies (R1),(R2) and (R3). Let $A, B \subseteq E$ be disjoint subsets such that $r(A \cup e) = r(A)$ for all $e \in B$. Then $r(A \cup B) = r(A)$.*

Proof. Exercise. Hint: Induction on $|B|$. □

Proof of Theorem 2.13. Let r be a function satisfying (R1), (R2) and (R3) and define $\mathcal{I} := \{I \subseteq E : r(I) = |I|\}$. We need to show that $(E, \mathcal{I})$ satisfies (I1), (I2) and (I3) and $r = r_M$.

Since $0 \leq r(\varnothing) \leq |\varnothing|$ we have $\varnothing \in \mathcal{I}$ and thus (I1) is true.

Let $I \in \mathcal{I}$ and fix $e \in I$. Then we have

$$|I| = r(I) = r((I \setminus e) \cup e) \leq r(I \setminus e) + r(e) - r(\varnothing) \leq r(I \setminus e) + 1$$

Hence $|I \setminus e| = |I| - 1 = r(I \setminus e)$ which implies $I \setminus e \in \mathcal{I}$ and so (I2) holds by induction.

For (I3) consider $I, J \in \mathcal{I}$ with $|I| < |J|$ and assume that $I \cup e \notin \mathcal{I}$ for all $e \in J \setminus I$. Hence $r(I \cup e) = r(I)$ for all $e \in J \setminus I$. The previous lemma now implies that $r(I \cup J) = r(I)$. But $|J| = r(J) \leq r(I \cup J) = r(I) = |I|$ yields a contradiction.

Thus $M = (E, \mathcal{I})$ is a matroid. Let $A \subseteq E$ and $I \subseteq A$ inclusion-maximal independent with $I \in \mathcal{I}$. Hence $r_M(A) = |I| = r(I) \leq r(A)$. Since I is inclusion-maximal, we have that $r(I \cup e) = r(I)$ for all $e \in A \setminus I$. The previous lemma now again implies that $r(I) = r(A)$ and hence $r_M(A) = r(A)$ for all A. □

3. (Broken) Circuits and the Characteristic Polynomial

We now turn to the dependent sets. Recall that we call the sets in $\mathcal{I}$ independent. We call the sets not contained in $\mathcal{I}$ **dependent**.

Definition 3.1. A minimal dependent set $C \subseteq E$ is a **circuit** of $(E, \mathcal{I})$, that is C is dependent but $C \setminus e$ is independent for all $e \in C$.

The term circuits stems from graphic matroids, here they are the cycles or circuits of the graph.

The collection of circuits completely describes the matroid, as the dependent sets are all sets containing a circuit and the independent sets are the sets that are not dependent. We can moreover define matroids via their circuits.

THEOREM 3.2. *Let E be a finite set and $\mathcal{C} \subseteq \mathcal{P}(E)$ a collection of subsets. Then $\mathcal{C}$ is the collection of circuits of a matroid if and only if it satisfies*

(C1) $\varnothing$ is not a circuit,
(C2) A proper subset of a circuit is not a circuit and
(C3) If $C_1, C_2 \in \mathcal{C}$ are distinct and $e \in C_1 \cap C_2$ then there exists a circuit $C_3 \subseteq (C_1 \cup C_2) \setminus e$.

We now generalize the definition of the characteristic polynomial of an arrangement to matroids:

Definition 3.3. Let M be a matroid of rank r and let $L(M)$ be its lattice of flats. Set $\mathrm{Int}(L(M)) := \{[X, Y] : X \leq Y\}$ to be the set of intervals. We define a more general version of the Möbius function $\mu : \mathrm{Int}(L(M)) \to \mathbb{Z}$ by setting

$$\begin{aligned} \mu(X, X) &= 1 \quad \text{for all } X \in L(M) \\ \mu(X, Y) &= -\sum_{X \leq Z < Y} \mu(X, Z) \quad \text{for all } X < Y. \end{aligned}$$

We moreover set $\mu(X) := \mu(\hat{0}, X)$.

We define the **characteristic polynomial** $\chi_M(t) \in \mathbb{Z}[t]$ of M to be

$$\chi_M(t) = \sum_{X \in L(M)} \mu(X) t^{r - r(X)}.$$

It follows from the definitions that this polynomial "essentially" agrees with the one of arrangements defined before:

Corollary 3.4. *Let $\mathcal{A}$ be a (central) arrangement and set $k := \dim\left(\bigcap_{H \in \mathcal{A}} H\right)$. Then*

$$\chi_{\mathcal{A}}(t) = t^k \chi_{M(\mathcal{A})}(t).$$

In particular, the polynomials agree for essential arrangements.

We consider the matroid M depicted in Figure 6 together with its lattice of flats as a running example in this section. We compute

$$\chi_M(t) = t^3 - 5t^2 + 8t - 4.$$

We first observe that the sign of the coefficients alternates. This is no coincidence:

THEOREM 3.5. *Let L be a geometric lattice and consider $X \leq Y$ in L. Then*

$$(-1)^{r(Y) - r(X)} \mu(X, Y) > 0.$$

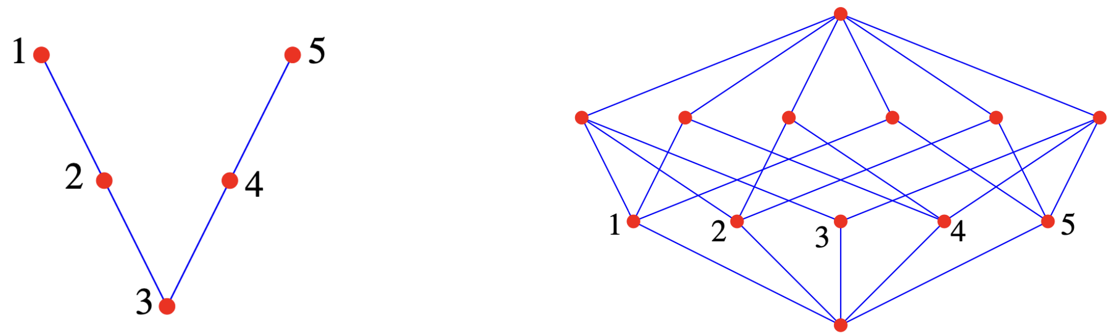


FIGURE 6. A rank 3 matroid on 5 elements with its lattice of flats.

PROOF. Using an induction it is enough to show the claim for $\hat{0} \leq \hat{1}$. This follows from a separate lemma that shows that for any atom A it holds that

$$\mu(\hat{0}, \hat{1}) = - \sum_{A \not\leq X <: \hat{1}} \mu(\hat{0}, X). \qquad \square$$

Thus when writing

$$\chi_M(t) = w_0 t^r + w_1 t^{r-1} + \cdots + w_r$$

we know that $(-1)^i w_i > 0$. The numbers $w_0, w_1, \ldots, w_r$ are called **Whitney numbers of the first kind**.

What is a combinatorial interpretation of the numbers $w_0, w_1, \ldots, w_r$ for a general matroid M?

Let M be a matroid on the ground set $E = \{e_1, \ldots, e_n\}$ and assume that these elements are ordered $e_1 < e_2 < \cdots < e_n$.

Definition 3.6. A **broken circuit** of M is a set $C \setminus u$ where C is a circuit of M and u is the smallest element in C in the above order. The **broken circuit complex** $BC(M)$ is defined by

$$BC(M) = \{S \subset E : S \text{ contains no broken circuit}\}.$$

In our running example, the matroid on the set $E = \{1, \ldots, 5\}$ (ordered in this way) has the circuits $123, 345, 1245$. Hence it has the broken circuits $23, 45, 245$. Its broken circuit complex is

$\varnothing$

1 2 3 4 5

12 13 14 15 24 25 34 35

124 125 134 135.

Note that $BC(M)$ is a simplicial complex, i.e., it is closed under taking subsets. For any simplicial complex Δ we denote by $f_i(\Delta)$ the number of i-dimensional faces (faces of cardinality $i+1$). Thus in our example

$$f_{-1}(BC(M)) = 1 \quad f_0(BC(M)) = 5 \quad f_1(BC(M)) = 8 \quad f_2(BC(M)) = 4.$$

Compare this with

$$\chi_M(t) = t^3 - 5t^2 + 8t - 4.$$

The rest of this lecture is devoted to explaining this "numerical coincidence" in general. For this we need some more machinery, that is useful in many places in combinatorics:

Let P be finite graded poset with $\hat{0}$ and $\hat{1}$. Let $\mathrm{Cov}(P) \subset P \times P$ be the set of cover relations in P.

Definition 3.7. An **R-labeling** of P is a map $\lambda : \mathrm{Cov}(P) \to \mathbb{Z}$ such that for every $X < Y$ in P there exists a unique saturated chain $X = X_0 <: X_1 <: \cdots <: X_k = Y$ such that

$$\lambda(X_0, X_1) \leq \lambda(X_1, X_2) \leq \cdots \leq \lambda(X_{k-1}, X_k).$$

Such a saturated chain is called an **increasing chain**.

EXAMPLE

THEOREM 3.8. *Let P be a finite graded poset with an R-labeling λ and $X \leq Y$ in P. Then $(-1)^{r(Y)-r(X)}\mu(X,Y)$ is the number of saturated chains $X = X_0 <: X_1 <: \cdots <: X_k = Y$ such that*

$$\lambda(X_0, X_1) > \lambda(X_1, X_2) > \cdots > \lambda(X_{k-1}, X_k).$$

There is a nice construction of an R-labeling for every lattice of flats of a matroid: Let M be a simple matroid with (ordered) ground set $E = \{e_1, \ldots, e_n\}$ and lattice of flats $L(M)$. For a cover relation $F <: F'$ define

$$\lambda(F, F') := \min(F' \setminus F).$$

EXAMPLE

Proposition 3.9. *This map λ is an R-labeling.*

PROOF. Let $F \subset F'$ be two flats. We show that there is a unique increasing chain from F to F' by induction on $k = r(F') - r(F)$. If $k = 1$, $F <: F'$ is a cover relation and the claim holds.

Now assume $k > 1$ and pick $e = \min(F' \setminus F)$. Set F_1 to be the smallest flat containing $F \cup e$, i.e., the closure of $F \cup e$ in M. Hence $F <: F_1$ is a cover relation with $\lambda(F, F_1) = e$. By induction there is an increasing chain $F_1 <: F_2 <: \cdots <: F_k = F'$. Since $\lambda(F_i, F_{i+1}) \in F' \setminus F_1 \subset F' \setminus (F \cup e)$, we have $\lambda(F_i, F_{i+1}) > e$ for all $i = 1, \ldots, k-1$. Thus by setting $F_0 = F$ we get an increasing chain $F_0 <: F_1 <: \cdots <: F_k$.

To prove that the chain is unique consider another increasing chain $F = F'_0 <: \cdots <: F'_k = F'$. By induction we have $F_1 \neq F'_1$ and hence $\lambda(F, F'_1) = f > e$. Thus $e \in F' \setminus F'_1$ and $e \notin F'_1$. So we can pick a maximal index j with $e \notin F'_j$ but $e \in F'_{j+1}$. Hence $\lambda(F_j)', F'_{j+1}) = e > f$ which contradicts the assumption that this chain is increasing. □

The next proposition establishes a connection between the broken circuit complex and the R-labeling of a simple matroid.

Proposition 3.10. *Let $\mathcal{C} = \{\varnothing = F_0 <: F_1 <: \cdots <: F_k\}$ be a saturated chain with $\lambda(\mathcal{C}) = (a_1, a_2, \ldots, a_k)$. Then $\{a_1, \ldots, a_k\} \in BC(M)$.*

PROOF. Assume $S = \{a_1, \ldots, a_k\}$ contains the broken circuit $C \setminus \min(C)$. Set $b = \min(C)$. This assumption implies $C \subseteq F_k$ and therefore there is a minimal j such that $C \subseteq F_j$. Hence $C \setminus b$ is not a subset of F_{j-1}. By construction we have $a_j = \lambda(F_{j-1}, F_j) = \min(F_j \setminus F_{j-1})$. Thus this implies $a_j, b \in F_j \setminus F_{j-1}$ and $a_j, b \in C$. As $b \neq a_j$ this is however a contradiction as both elements were supposed to be the minimum of both of these sets. □

Now we can conclude with a proof that the coefficients of the characteristic polynomial agree up to sign with the f-vector of the broken circuit complex:

THEOREM 3.11. *Let M be a simple matroid of rank r with characteristic polynomial $\chi_M(t) = w_0 t^r + w_1 t^{r-1} + \cdots + w_r$. Then*

$$(-1)^i w_i = f_{i-1}(BC(M)) = |\{S \in BC(M) : |S| = i\}|.$$

PROOF. For rank k we have $(-1)^k w_k = \sum_F (-1)^k \mu(\varnothing, F)$ where the sum is over all flats of rank k. Theorem 3.8 implies that $(-1)^k \mu(\varnothing, F)$ is the number of saturated chains $\mathcal{C} = \{\varnothing = F_0 <: F_1 <: \ldots, <: F_k = F\}$ for which $\lambda(\mathcal{C}) = (a_1 > a_2 > \cdots > a_k)$ is decreasing. The previous proposition shows that $\{a_1, a_2, \ldots, a_k\} \in BC(M)$. So the only thing left to show is that any set $S \in BC(M)$ indeed comes from such a decreasing chain.

Let $S \in BC(M)$ and suppose $S = \{a_1 > a_2 \cdots > a_k\}$. Define F_i to be the smallest flat containing $\{a_1, \ldots, a_i\}$, again the closure, for $i = 0, \ldots, k$. This is a saturated chain of flats as S is independent. Set $\lambda(F_i, F_{i+1}) = \min(F_{i+1} \setminus F_i) = b$ and assume $b \neq a_{i+1}$. Since $a_1, \ldots, a_{i+1}$ is independent, there is a circuit

$C \subseteq \{a_1, \ldots, a_{i+1}, b\}$. Since $b < a_{i+1}$, we have $\min(C) = b$. Hence $C \setminus \min(C) \subseteq \{a_1, \ldots, a_{i+1}\} \subseteq S$ which contradicts $S \in BC(M)$. □