

# Finite reflection groups, hyperplane arrangements, and (oriented) matroids

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# Contents

1. Introduction	4
Chapter 1. Finite reflection groups	5
1. The symmetric group	5
2. Reflections and reflection groups	6
3. Root systems	8
4. Reflection arrangements and the length function	12
5. Fundamental domains and parabolic subgroups	16
6. The Coxeter complex	20
7. Classification of finite reflection groups	25
Chapter 2. Arrangements and simpliciality	29
1. Arrangements of hyperplanes	29
2. Simplicial arrangements	31
Bibliography	33
Index	35

**1. Introduction**

To be written...

## Finite reflection groups

### 1. The symmetric group

The **symmetric group**  $\mathfrak{S}_n$  is set of bijections from  $[n] := \{1, 2, \dots, n\}$  to itself. We can all agree that permutations and their combinatorics are ubiquitous throughout all of mathematics. Here we recall the basic properties of  $\mathfrak{S}_n$  and view it as a group generated by reflections in hyperplanes.

We represent a permutation  $\sigma : [n] \rightarrow [n]$  in **one-line** notation, which is the bottom row of the table

$$\begin{array}{c|cccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline \sigma(n) & 4 & 6 & 7 & 1 & 9 & 3 & 5 & 2 & 8 \end{array}$$

Thus  $\sigma = 467193528$ . The set of bijections becomes a group under composition. So, for example, for  $\tau = 581639274$ , the product with  $\sigma$  is  $\sigma\tau = 924378651$ . Alternatively, we write  $\sigma$  in **cycle notation**  $\sigma = (14)(2637598)$ . Fixed points, i.e., cycles of length 1 are usually omitted.

A **transposition** is a permutation that swaps two numbers  $i$  and  $j$  and leave all others fixed. We denote such a transposition by  $(i, j)$ . Note that  $(i, j)\sigma$  swaps the *numbers*  $i$  and  $j$  in the one-line notation, while  $\sigma(i, j)$  swaps the entries at *positions*  $i$  and  $j$ .

$$\begin{aligned} (7, 9)\sigma &= 469173528 \\ \sigma(7, 9) &= 467193825 \end{aligned}$$


Also note that  $\sigma(i, j) = (\sigma(i), \sigma(j))\sigma$  for all  $i < j$ .

The symmetric group is generated by **transpositions**. In fact,  $\mathfrak{S}_n$  is generated by **adjacent** (or **simple**) transpositions, that is, the  $n - 1$  transpositions of the form  $s_i = (i, i + 1)$  for  $i = 1, \dots, n - 1$ . For example,

$$\begin{aligned} \sigma &= s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_6 s_5 s_4 s_3 s_8 s_7 s_6 s_5 s_6 s_7 \\ &= (3, 4)(2, 3)(1, 2)(5, 6)(4, 5)(3, 4)(2, 3)(6, 7)(5, 6)(4, 5)(3, 4)(8, 9)(7, 8)(6, 7)(5, 6)(6, 7)(7, 8) \end{aligned}$$

This is a **reduced expression**, in the sense that it uses the minimal number of simple transpositions. Reduced expressions are in general not unique. For  $\sigma$  there are 5630196 many distinct reduced expressions. The number of simple transpositions used is independent of the reduced expression and called the **length**  $\ell(\sigma)$ . An **inversion** of  $\sigma$  is a pair  $(i, j)$  with  $1 \leq i < j \leq n$  and  $\sigma(i) > \sigma(j)$ . The **inversion number**  $\text{inv}(\sigma)$  is the number of inversions.

**Exercise 1.1.** Let  $\sigma \in \mathfrak{S}_n$  be a permutation.

- (1) Show that  $\ell(\sigma) = \text{inv}(\sigma)$ .
- (2) Find a procedure to generate all reduced expressions. 

We can view permutations as acting on  $\mathbb{R}^n$ . For  $\sigma \in \mathfrak{S}_n$ , define  $P_\sigma \in \mathbb{R}^{n \times n}$  by

$$(P_\sigma)_{i,j} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases}$$

If  $e_1, \dots, e_n$  are the standard basis vectors of  $\mathbb{R}^n$ , then  $P_\sigma e_i = e_{\sigma(i)}$  and hence  $P_\sigma P_\tau = P_{\sigma\tau}$ . Note that  $(P_\sigma)^{-1} = P_{\sigma^{-1}} = P_\sigma^t$  and hence  $P_\sigma$  is an *orthogonal* matrix with respect to the standard inner product

$\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n$  on  $\mathbb{R}^n$ . So, for our  $\sigma$  we get

$$P_\sigma P_\tau = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = P_{\sigma\tau}$$

Note that for a transposition  $(i, j)$ ,  $P_{(i,j)}v = v$  if and only if  $v_i = v_j$ . That is,  $P_{(i,j)}$  acts as the identity on the linear hyperplane  $H_{ij} = \{v \in \mathbb{R}^n : v_i = v_j\}$ . Moreover,  $P_{(i,j)}$  is the reflection in the hyperplane  $H_{ij}$  in the sense that  $v - P_{(i,j)}(v)$  is perpendicular to  $H_{ij}$ . Thus, we can view  $\mathfrak{S}_n$  as a finite group of linear transformations that is generated by reflections in hyperplanes.

As a sneak peek, let us see how the geometry of the arrangement of the  $\binom{n}{2}$  hyperplanes  $H_{i,j}$  reflects properties of  $\mathfrak{S}_n$ . A point  $v \in \mathbb{R}^n$  is not contained in any of the hyperplanes  $H_{i,j}$  if and only if  $v_i \neq v_j$  for all  $i < j$ . Hence, there is a unique permutation  $\sigma$  such that

$$v_{\sigma(1)} < v_{\sigma(2)} < \cdots < v_{\sigma(n)}.$$

Moreover,  $u$  is in the same connected component of  $\mathbb{R}^n \setminus \bigcup_{i < j} H_{i,j}$  if and only if  $u$  is contained in the open convex set

$$C_\sigma = \{v \in \mathbb{R}^n : v_{\sigma(1)} < v_{\sigma(2)} < \cdots < v_{\sigma(n)}\}.$$

Thus

$$\mathbb{R}^n \setminus \bigcup_{i < j} H_{i,j} = \bigsqcup_{\sigma \in \mathfrak{S}_n} C_\sigma$$

and since  $C_\sigma = P_\sigma(C_{\text{id}})$ , any two sets  $C_\sigma$  are isometric.

## 2. Reflections and reflection groups

Let  $V$  be an  $n$  dimensional real vector space with inner product  $\langle \cdot, \cdot \rangle$ . A **linear hyperplane** is a linear subspace of dimension  $n - 1$ . That is,  $H$  is of the form

$$H = H_\alpha = \{v \in V : \langle \alpha, v \rangle = 0\}$$

for some  $\alpha \in \mathbb{R}^n \setminus 0$ . Note that  $\alpha$  is unique up to scaling. The (orthogonal) **reflection** in  $H$  is the linear transformation  $s_H : V \rightarrow V$  such that  $s_H(v) = v$  for all  $v \in H$  and  $s_H(\alpha) = -\alpha$ . This allows us to give an explicit description of  $s_H$  as

$$s_H(v) = v - \frac{2\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle} \alpha. \quad (1)$$

This does not depend on the choice of  $\alpha$  but it is customary to write this as  $s_\alpha$ . In particular  $s_\alpha^2 = \text{id}$  and  $s_\alpha$  is an element of the orthogonal group  $O(V) = \{g \in \text{GL}(V) : \langle g(u), g(v) \rangle = \langle u, v \rangle \text{ for all } u, v \in V\}$ .

A **finite reflection group** is a finite group of linear transformations  $W \subseteq \text{GL}(V)$  that is generated by (orthogonal) reflections in hyperplanes.

If  $W$  is generated by a single reflection, then  $W = \{e, s_\alpha\}$ .

To get a feeling, let us consider the case that  $W$  is generated by two reflections  $s_\alpha$  and  $s_\beta$ . Now  $\alpha, \beta$  are linearly independent and span a 2-dimensional subspace  $L = \text{span}\{\alpha, \beta\}$ . Since  $V = L \oplus L^\perp$ , and  $s_\alpha, s_\beta$  restrict to the identity on  $L^\perp$ , it suffices to consider the case  $V = \mathbb{R}^2$ . More generally, if  $W$  is a reflection group, then the **fixed space** is  $V^W := \{u \in V : wu = u \text{ for all } w \in W\}$  and we can restrict the action of  $W$  to  $(V^W)^\perp$ . If  $V^W = \{0\}$ , then we call  $W$  **essential** (relative to  $V$ ). The **rank** of  $W$  is the dimension  $\dim(V^W)^\perp = \dim V - \dim V^W$ . Hence, if  $W$  is minimally generated by 2 reflections, then  $W$  is of rank 2.

Let  $\alpha, \beta \in \mathbb{R}^2 \setminus 0$ . Let  $\theta$  be the angle between  $H_\alpha$  and  $H_\beta$ . The angle satisfies  $\theta = \pi - \angle(\alpha, \beta) = \pi - \cos^{-1} \left( \frac{\langle \alpha, \beta \rangle}{\sqrt{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}} \right)$ . Figure 1 shows the situation. If  $v$  has angle  $\omega$  to the line  $H_\alpha$ , then  $\angle(v, s_\alpha(v)) = 2\omega$ .

Likewise, if the angle between  $s_\alpha(v)$  and  $H_\beta$  is  $\gamma$ , then  $\angle(s_\alpha(v), s_\beta(s_\alpha(v))) = 2\gamma$ . Since  $\omega + \gamma = \theta$ , we infer that  $\angle(v, s_\beta(s_\alpha(v))) = 2\omega + 2\gamma = 2\theta$ . Thus  $s_\beta s_\alpha$  is a rotation by  $2\theta$ .

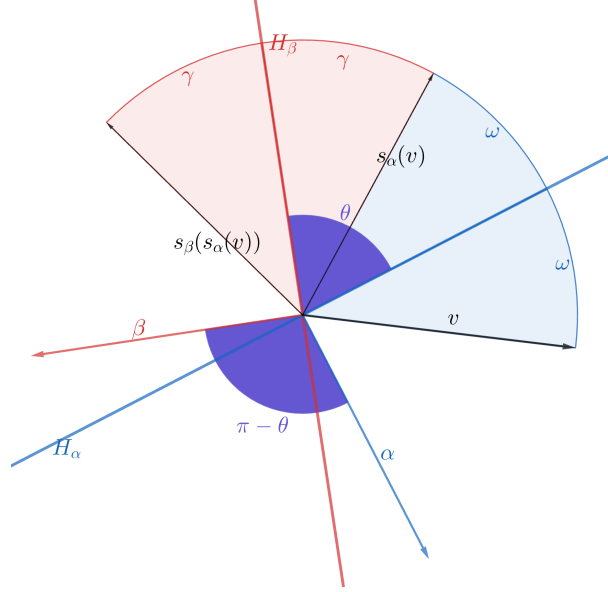


FIGURE 1. The composition of two reflections in the plane is a rotation.

Now if  $W$  is finite, then there is (a minimal)  $m \geq 2$  such that  $(s_\beta s_\alpha)^m = \text{id}$ . Such an  $m$  must satisfy  $m\theta = \ell\pi$  for some  $\ell \geq 1$ .

**Proposition 2.1.** *Let  $H_\alpha, H_\beta$  be hyperplanes with an angle  $\frac{k}{\ell}\pi$  between them. Let  $W$  be the reflection group generated by  $s_\alpha$  and  $s_\beta$ . Then there is a reflection  $s_\gamma \in W$  whose hyperplane  $H_\gamma$  has an angle of  $\frac{1}{\ell}\pi$  to  $H_\alpha$  and  $s_\alpha, s_\gamma$  generate  $W$ .*

PROOF. We can assume that  $k$  and  $\ell$  are coprime and hence there is an  $r > 0$  such that  $rk = q\ell + 1$  for some  $q$ . Thus  $w := (s_\beta s_\alpha)^r$  is a rotation by  $\frac{2}{\ell}\pi$ . Consider  $w' := ws_\alpha \in W$ . This is a reflection (check this!) in some hyperplane  $H_\gamma$ , i.e.,  $w' = s_\gamma$  and  $w's_\alpha$  is a rotation of  $\frac{2}{\ell}\pi$ . Thus  $H_\gamma$  and  $H_\alpha$  have an angle of  $\frac{1}{\ell}\pi$  and  $s_\beta s_\alpha = (s_\gamma s_\alpha)^k$ , that is,  $s_\beta = (s_\gamma s_\alpha)^k s_\alpha = (s_\gamma s_\alpha)^{k-1} s_\gamma$ .  $\square$

We can always assume that  $W$  is generated by reflections in hyperplanes with angle  $\frac{\pi}{m}$  between them.

If  $m = 2$ , then  $\alpha \perp \beta$ , that is,  $\alpha \in H_\beta$  and  $\beta \in H_\alpha$ . Hence  $s_\alpha s_\beta = s_\beta s_\alpha$  and  $W = \{\text{id}, s_\alpha, s_\beta\}$ .

If  $m \geq 3$ , then pick  $u_1 \in H_\alpha \setminus \{0\}$ . The orbit  $Wu_1 = \{wu_1 : w \in W\}$  will consist of  $m$  points  $u_1, u_2, \dots, u_m$  (why?  $\textcircled{S}$ ), which are the vertices of a regular polygon  $Q_m$  (why??  $\textcircled{S}$ ). The group  $W$  is the symmetry group of  $Q_m$ , called the **dihedral group**, denoted by  $\mathcal{D}_m$ . The number of elements of  $\mathcal{D}_m$  is  $2m$ .

**Exercise 2.2.** Answer the two questions marked  $\textcircled{S}$  in the paragraph above.  $\textcircled{S}$

This gives a complete classification of rank-2 reflection groups.


Notice that for any  $m \geq 2$  if  $u$  is not contained in any of the reflection hyperplanes of  $W$ , then  $Wu$  are the vertices of a  $2m$ -gon that is not necessarily regular and for which  $W$  is not the symmetry group. We'll get back to these polygon much later.

Generally, if  $W$  is a finite reflection group and  $s_\alpha, s_\beta \in W$  are two reflections, then the subgroup  $W' = \langle s_\alpha, s_\beta \rangle$  generated by  $s_\alpha, s_\beta$  is a finite reflection group of rank 2. Let us consider the various such subgroups for the symmetric group  $\mathfrak{S}_n$ . A normal for the hyperplane  $H_{i,j}$  for  $1 \leq i < j \leq n$  is given by  $\alpha_{i,j} := e_j - e_i$ . Thus, for two distinct transpositions  $(i, j), (k, l)$ , we have  $\alpha_{i,j} \perp \alpha_{k,l}$  iff  $\{i, j\} \cap \{k, l\} = \emptyset$ . Otherwise


$|\{i, j\} \cap \{k, l\}| = 1$  and the angle between  $H_{(i,j)}$  and  $H_{(k,l)}$  is  $\frac{\pi}{3}$ . In particular,  $\mathfrak{S}_3$  is the symmetry group of the regular triangle embedded in  $\mathbb{R}^3$  with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ .

The symmetric group  $\mathfrak{S}_n$  fixes the linear subspace  $\{(t, t, \dots, t) : t \in \mathbb{R}\}$  pointwise and is of rank  $n - 1$ . In light of the classification of finite reflection groups that we will be seeing in a few lectures, let us start by calling  $\mathfrak{S}_n$  a reflection group of **type**  $A_{n-1}$ .


We close this first lecture with introducing one more example. For  $i = 1, \dots, n$ , let  $z_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be reflection in the hyperplane  $H_i = \{v \in \mathbb{R}^n : v_i = 0\}$ . That is,  $z_i$  flips the sign of the  $i$ -th coordinate of  $v$ . Consider the group  $W$  that is generated by  $s_{(i,j)}$  for  $1 \leq i < j \leq n$  as well as  $z_1, \dots, z_n$ . This is a group generated by reflections but it's not obvious that  $W$  is finite. To see that  $W$  is finite, consider the set  $S = \{-1, +1\}^n$  of  $\pm 1$ -vectors of length  $n$ . The subgroup  $G \subseteq \text{GL}(\mathbb{R}^n)$  of linear symmetries of  $S$  is finite (why??) and  $W$  is clearly a subgroup. That makes  $W$  finite as well.


**Exercise 2.3.** Let  $S \subset \mathbb{R}^n$  a finite set such that  $\text{span}(S) = \mathbb{R}^n$ . Let  $G$  be the group of linear transformations  $g \in \text{GL}(\mathbb{R}^n)$  such that  $gS = S$ . Show that  $G$  is a finite group. 

Note that  $W$  contains more reflections than the ones we used to generate. Indeed,  $z_i s_{(i,j)} z_i$  maps  $e_i$  to  $-e_j$  and  $e_j$  to  $-e_i$  and leaves  $e_k$  fixed for  $k \notin \{i, j\}$ . But this means it fixes all points of the hyperplane  $H = \{v \in \mathbb{R}^n : v_i = -v_j\}$  and maps its normal  $e_i + e_j$  to its negative. The finite reflection group  $W$  is of rank  $n$  and we say that it is of type  $\mathbf{B}_n$ .

**Exercise 2.4.** Show that the reflection group of type  $B_n$  is the symmetry group of the  $n$ -dimensional cube  $C_n = \{v \in \mathbb{R}^n : |v_i| \leq 1 \text{ for all } i = 1, \dots, n\}$ . 

**Exercise 2.5.** A convex polytope  $P$  in  $\mathbb{R}^3$  is the inclusion-minimal convex set containing a given finite set of points. The symmetry group  $G$  of  $P$  is the group of linear transformations  $g \in \text{GL}(\mathbb{R}^3)$  such that  $gP = P$ .

The boundary of  $P$  consists of vertices, edges, and faces. A **flag** of  $P$  is a choice of a vertex  $v$ , an edge  $e$ , and a face  $F$  such that  $v \in e \subset F$ . A convex polytope  $P$  is called **regular** if for two flags  $v \in e \subset F$  and  $v' \in e' \subset F'$  there is  $g \in G$  such that  $gv = v'$ ,  $ge = e'$ , and  $gF = F'$ . Show that if  $P$  is regular, then  $G$  is a finite reflection group. 

**Exercise 2.6.** Consider two infinitely long walls meeting in a corner at an angle  $\alpha \in (0, \pi]$ . Show that any kicked ball (which doesn't lose momentum) can meet the walls only a finite number of times. What is the maximal number of times a ball can hit the walls? 

### 3. Root systems

Let  $W$  be a finite reflection group acting on  $V$  and let  $\mathcal{A}(W)$  be the collection of reflecting hyperplanes of  $W$ . If  $H_\alpha \in \mathcal{A}(W)$  and  $w \in O(V)$ , then  $ws_\alpha w^{-1}$  sends  $w\alpha$  to its negative and pointwise fixes  $wH_\alpha = \{v \in V : \langle \alpha, w^{-1}v \rangle = \langle w\alpha, v \rangle = 0\} = H_{w\alpha}$ . Here we used the fact that  $w \in O(V)$  and hence  $w^* = w^{-1}$ . If  $w \in W$ , then  $ws_\alpha w^{-1} \in W$ , we infer that  $H_{w\alpha} \in \mathcal{A}(W)$ . Hence  $w\mathcal{A}(W) = \mathcal{A}(W)$  for all  $w \in W$ . Instead of the action of  $W$  on the reflection hyperplanes, it is customary to consider the action on a collection of normal vectors to the reflection hyperplanes. Since  $s_\alpha(\alpha) = -\alpha$  it is not sufficient to pick a normal vector for each hyperplane.

A **root system**  $\Phi$  is a finite non-empty subset of  $V \setminus 0$  such that for every  $\alpha \in \Phi$

- (R1)  $\Phi \cap \text{span}(\alpha) = \{-\alpha, \alpha\}$ , and
- (R2)  $s_\alpha(\beta) \in \Phi$  for all  $\beta \in \Phi$ .

The elements of  $\Phi$  are called **roots**. The **rank** of  $\Phi$  is the dimension of  $\text{span}(\Phi)$ .


Every finite reflection group  $W$  gives rise to a (actually many) root system:


$$\Phi = \{\alpha \in V : \langle \alpha, \alpha \rangle = 1, s_\alpha \in W\}$$



Conversely, if  $\Phi$  is a root system, then define  $W$  as the subgroup of  $O(V)$  generated by the reflections  $\{s_\alpha : \alpha \in \Phi\}$ . We may assume that  $V = \text{span}(\Phi)$  and appealing to Exercise 2.5 shows that  $W$  is a finite group.

Clearly if  $\Phi$  is a root system, then  $t\Phi = \{t\alpha : \alpha \in \Phi\}$  is a root system as well for all  $t \in \mathbb{R} \setminus 0$ . We call  $\Phi$  **reducible** if there is a partition  $\Phi = \Phi' \uplus \Phi''$  such that  $\Phi' \perp \Phi''$  and  $\Phi'$  and  $\Phi''$  are root systems. If no partition exists, then  $\Phi$  is **irreducible**.


**Exercise 3.1.** Let  $\Phi$  be a root system. For any non-empty  $U \subseteq \Phi$ , show that  $\Phi \cap \text{span}(U)$  is a root system. 

**Exercise 3.2.** Let  $\Phi$  be an irreducible root system. Show that there are at most two different lengths of roots. (Hint: Show this first for root systems of rank 2.) 

**Example 3.3** (A root system for type  $A_{n-1}$  and type  $B_n$ ). For type  $A_{n-1}$ : The set  $\Phi = \{e_i - e_j : i, j \in [n], i \neq j\}$  is a root system for the symmetric group and since  $\mathfrak{S}_n$  acts transitively on  $\Phi$ ,  $\Phi$  is unique up to scaling.

For type  $B_n$ : The set  $\Phi = \{\pm(e_i - e_j), \pm(e_i + e_j) : 1 \leq i < j \leq n\} \cup \{\pm e_1, \dots, \pm e_n\}$  is a root system whose associated reflection group is that of type  $B_n$  (check this!). Note that the roots have lengths 1 and  $\sqrt{2}$ . Since  $W$  acts by orthogonal transformations, there is more than one orbit of  $\Phi$  under  $W$ .  $\diamond$

We call  $c \in V$  **generic** relative to  $\Phi$  if  $\langle c, \alpha \rangle \neq 0$  for all  $\alpha \in \Phi$ . Since  $\Phi$  is finite, generic  $c$ 's exist.

**Exercise 3.4.** Identifying  $V = \mathbb{R}^n$ , show that there are infinitely many  $t \in \mathbb{R}$  such that  $c = (1, t, t^2, \dots, t^n)$  is generic for  $\Phi$ . 

A **positive system** of  $\Phi$  is a subset  $\Phi^+ \subset \Phi$  of the form

$$\Phi^+ = \{\alpha \in \Phi : \langle c, \alpha \rangle > 0\}$$

for some generic  $c$ .

Clearly  $|\Phi^+| = \frac{1}{2}|\Phi|$ , as  $\Phi^+$  selects one element from each pair  $-\alpha, \alpha \in \Phi$ . But our choice serves a greater purpose. We seek to find smallest sets of reflections that generate  $W$ . As it will turn out, these subsets will be as small as possible.

**Example 3.5** (Positive systems for type  $A_{n-1}$  and  $B_n$ ). For type  $A_{n-1}$ : A vector  $c$  is generic relative to  $\Phi = \{e_i - e_j : i, j \in [n], i \neq j\}$  if and only if  $c_i \neq c_j$  for all  $i \neq j$ . Consider  $c = (c_1 < c_2 < \dots < c_n)$ . Then  $\Phi^+ = \{e_j - e_i : 1 \leq i < j \leq n\}$ .

For type  $B_n$ :  $c$  is generic if  $|c_i| \neq |c_j|$  for all  $i \neq j$  and  $c_i \neq 0$  for all  $i$ . For  $c = (0 < c_1 < \dots < c_n)$ , the positive system is  $\Phi^+ = \{e_j - e_i, e_i + e_j : i < j\} \cup \{e_1, \dots, e_n\}$ .  $\diamond$

For a finite set  $U = \{u_1, \dots, u_m\} \subset V$ , we define the **conical hull** as the set

$$\text{cone}(U) = \{\mu_1 u_1 + \mu_2 u_2 + \dots + \mu_m u_m : \mu_1, \dots, \mu_m \geq 0\}.$$

We call a finite set  $U \subset V$  **acyclic** if there is  $c \in V$  with  $\langle c, u \rangle > 0$  for all  $u \in U$ .

**THEOREM 3.6.** *Let  $U \subset V$  be acyclic. If  $U \cap \text{span}(u) = \{u\}$  for all  $u \in U$ , then there is a unique inclusion-minimal  $G \subseteq U$  with  $\text{cone}(G) = \text{cone}(U)$ .*

**PROOF.** Let  $G = \{g_1, \dots, g_k\} \subset U$  be an inclusion-minimal generating set for  $C := \text{cone}(U)$ . That is,  $C = \text{cone}(G)$  but  $C \neq \text{cone}(G \setminus g_j)$  for all  $j = 1, \dots, k$ . Assume that  $H = \{h_1, \dots, h_l\} \subset U$  also generates  $C$  but, say,  $g_1 \notin H$ . Then there are  $a_1, \dots, a_l \geq 0$  such that  $g_1 = \sum_i a_i h_i$  at least two  $a_i$  nonzero. On the other hand, there are  $b_{ij} \geq 0$  such that  $h_i = \sum_j b_{ij} g_j$ . Setting  $d_j = \sum_i a_i b_{ij} \geq 0$ , we get

$$g_1 = d_1 g_1 + d_2 g_2 + \dots + d_k g_k.$$

Said differently,  $(1-d_1)g_1$  is contained in  $\text{cone}(G \setminus g_1)$ . We cannot have  $1-d_1 > 0$ , as this would contradict the assumption that  $G$  is inclusion-minimal. If  $1-d_1 < 0$ , then

$$0 > (1-d_1)\langle c, g_1 \rangle = \sum_{i=2}^k d_i \langle c, g_i \rangle \geq 0.$$

This leaves  $d_1 = 1$ . But the same calculation then shows  $d_2 = \dots = d_k = 0$ , which implies  $g_1 \in S$ . Hence  $G \subseteq U$ .  $\square$

By Theorem 3.6, there is a unique inclusion-minimal subset  $\Delta \subseteq \Phi^+$  that minimally generates  $\text{cone}(\Phi^+)$ . We call  $\Delta$  a **simple system**. Figure 2 shows two examples of root systems, positive systems, and simple systems in rank 2.

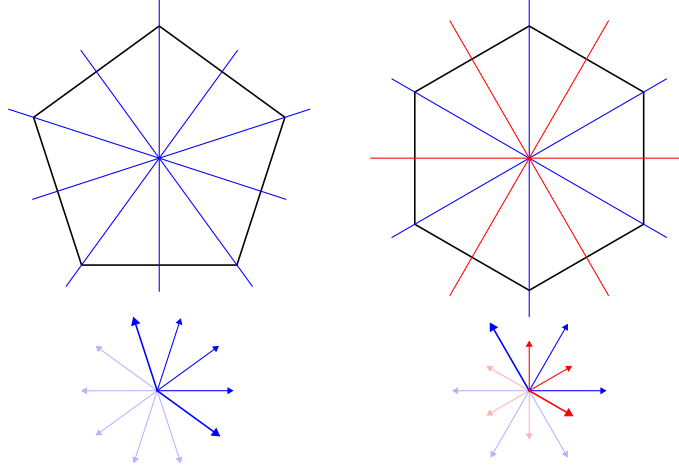


FIGURE 2. Two examples of finite reflection groups in the plane. Below root systems are shown. The positive system is opaque. The simple roots are slightly thicker.

We start with a seemingly technical but important lemma about simple systems.

**Lemma 3.7.** *Let  $\Delta$  be a simple system. Then  $\langle \alpha, \beta \rangle \leq 0$  for all  $\alpha, \beta \in \Delta, \alpha \neq \beta$ .*

PROOF. Let  $\alpha, \beta \in \Delta$  with  $\alpha \neq \beta$ . Then  $L = \text{span}(\alpha, \beta)$  is a 2-dimensional subspace and Exercise 3.1 yields that  $\Phi \cap L$  is a root system with positive system  $\Phi^+ \cap L$ . Let  $\Delta' \subseteq \Phi^+ \cap L$  be a simple system. If  $\alpha \notin \Delta'$ , then  $\Delta' \cup (\Delta \setminus \alpha)$  generates  $\text{cone}(\Phi^+)$ , contradicting Theorem 3.6. Hence  $\alpha, \beta \in \Delta'$  and, using that  $L$  is 2-dimensional,  $\Delta' = \{\alpha, \beta\}$ . In particular,  $\alpha, \beta$  are linearly independent.

Now assume that  $\langle \alpha, \beta \rangle > 0$ . We know that  $\gamma = s_\alpha(\beta)$  is contained in  $\Phi \cap L$ . Now  $s_\alpha(\beta) = \beta + \mu\alpha$ , where  $\mu = \frac{-2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} < 0$ . But  $\gamma$  or  $-\gamma$  is contained in  $\Phi^+ \cap L$  but the unique expression of  $\gamma$  as a linear combination of  $\alpha$  and  $\beta$  has positive and negative coefficients. This contradicts that  $\Phi^+ \cap L \subseteq \text{cone}(\alpha, \beta)$ .  $\square$

A main consequence of this is the following.

**Corollary 3.8.** *Let  $\Delta$  be a simple system, then  $\Delta$  is linear independent. In particular, every root  $\alpha \in \Phi$  has a unique expression in terms of simple roots, with all coefficients nonnegative or nonpositive.*

PROOF. Let  $\alpha_1, \dots, \alpha_m \in \Delta$  be a minimally dependent subset. Then there are  $a_1, \dots, a_m \in \mathbb{R}$  not all zero such that  $0 = \sum_i a_i \alpha_i$ . Since  $\langle c, \alpha_i \rangle > 0$  for some generic  $c$  that brought us  $\Phi^+$ , not all  $a_i$  are of the same sign. We may assume that  $a_1, \dots, a_k < 0 < a_{k+1}, \dots, a_m$ . Then

$$-\sum_{i=1}^k a_i \alpha_i = \sum_{j=k+1}^m a_j \alpha_j =: v$$

and minimality yields  $v \neq 0$ . We compute

$$0 < \langle v, v \rangle = \sum_{i=1}^k \sum_{j=k+1}^m (-\alpha_i) \alpha_j \langle \alpha_i, \alpha_j \rangle \leq 0,$$

which is a contradiction.  $\square$

The corollary furnishes another characterization of simple systems.

**Corollary 3.9.** *Let  $S \subseteq \Phi$  be a basis for  $\text{span}(\Phi)$  such that  $\Phi \subset \text{cone}(S) \cup \text{cone}(-S)$ , then  $S$  is a simple system.*

PROOF. We may assume that  $\text{span}(\Phi) = V$ . Then  $S = \{\alpha_1, \dots, \alpha_n\}$  is a basis and there is some  $c \in V$  with  $\langle c, \alpha_i \rangle > 0$  for all  $i$ . As every  $\beta \in \Phi$  is of the form  $\beta = \sum_i a_i \alpha_i$  for some  $\alpha_i$  that are all nonnegative or nonpositive, this shows that  $c$  is generic. In particular  $\beta \in \Phi^+$  if and only if  $\beta \in \text{cone}(S) \cap \Phi$ , which shows that  $S$  is a simple system.  $\square$

**Example 3.10** (Simple systems for types  $A_{n-1}$  and  $B_n$ ). Continuing Example 3.5, let us choose  $c = (0 < c_1 < \dots < c_n)$ , then the positive system for type  $A_{n-1}$  is  $\Delta = \{e_2 - e_1, e_3 - e_2, \dots, e_n - e_{n-1}\}$ . Indeed, for any  $e_j - e_i \in \Phi^+$  with  $1 \leq i < j \leq n$  we have

$$e_j - e_i = (e_j - e_{j-1}) + (e_{j-1} - e_{j-2}) + \dots + (e_{i+1} - e_i).$$

Likewise, for type  $B_n$ , we obtain  $\Delta = \{e_1, e_2 - e_1, e_3 - e_2, \dots, e_n - e_{n-1}\}$ . In addition to the positive roots  $e_j - e_i$  for  $i < j$ , we get  $e_1$  and  $e_j = e_j - e_{j-1} + e_{j-1}$  by induction on  $j$ . The positive roots  $e_j + e_i$  are then obvious.  $\diamond$

The fact that every root has a unique representation in terms of  $\Delta$  with nonzero coefficients of like sign is the key in showing that  $\Delta$  yields a generating set for  $W$ .

**Lemma 3.11.** *Let  $\Delta \subseteq \Phi^+ \subseteq \Phi$ . For  $\alpha \in \Delta$  and  $\beta \in \Phi^+$ , we have  $s_\alpha(\beta) \in \Phi^+$  if  $\beta \neq \alpha$  and  $s_\alpha(\beta) = -\alpha$  otherwise.*

PROOF. Let  $\Delta = \{\alpha_1, \dots, \alpha_n\}$ ,  $\alpha = \alpha_1$ , and  $\beta = \sum_i a_i \alpha_i$  with  $a_1, \dots, a_n \geq 0$ . If  $\alpha \neq \beta$ , then there is some  $a_i > 0$  for  $i \geq 2$ . Now

$$s_\alpha(\beta) = \beta - \mu \alpha_1 = (a_1 - \mu) \alpha_1 + \sum_{i \geq 2} a_i \alpha_i.$$

Since  $s_\alpha(\beta) \in \Phi$  and the expression of  $s_\alpha(\beta)$  is unique with nonzero coefficients of like sign, the fact that  $\alpha_i > 0$ , forces  $a_1 - \mu$  to be positive as well.  $\square$

**Theorem 3.12.** *Let  $W$  be a finite reflection group with root system  $\Phi$ . Let  $\Delta \subset \Phi$  be a simple system, then  $\{s_\alpha : \alpha \in \Delta\}$  generates  $W$*

PROOF. Let  $W'$  be the subgroup of  $W$  generated by the reflections  $\{s_\alpha : \alpha \in \Delta\}$ . It suffices to show that for every  $\beta \in \Phi^+$ , there is  $w \in W'$  such that  $w\beta \in \Delta$ . Indeed if  $\beta = w^{-1}\alpha$  for  $w \in W'$  and  $\alpha \in \Delta$ , then  $s_\beta = s_{w^{-1}\alpha} = w^{-1}s_\alpha w$  and hence  $s_\beta \in W'$ .

For any  $\gamma \in \Phi^+$  let  $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$  be the unique representation with all  $c_\alpha \geq 0$  and define the **height** as  $\text{ht}(\beta) = \sum_{\alpha \in \Delta} c_\alpha$ . Pick  $\gamma \in W'\beta \cap \Phi^+$  with  $\text{ht}(\gamma)$  minimal. We claim that  $\gamma \in \Delta$ .

Since  $\gamma \neq 0$ , we have

$$0 < \langle \gamma, \gamma \rangle = \sum_{\alpha \in \Delta} c_\alpha \langle \gamma, \alpha \rangle$$

and thus there is some  $\alpha \in \Delta$  with  $c_\alpha > 0$  and  $\langle \gamma, \alpha \rangle > 0$ . But by (1),  $s_\alpha(\gamma) = \gamma - \mu\alpha$ , where  $\mu > 0$ . If  $s_\alpha(\gamma)$  is in  $\Phi^+$ , then  $\text{ht}(s_\alpha(\gamma)) < \text{ht}(\gamma)$ , which would contradict our choice of  $\gamma$ . Hence  $s_\alpha(\gamma) \in -\Phi^+$  and by Lemma 3.11, this implies  $\gamma = \alpha$ .  $\square$

**Example 3.13** (Generation by simple reflections in types  $A_{n-1}$  and  $B_n$ ). Continuing Example 3.10, note that  $\text{ht}(e_j - e_i) = j - i$ . Applying, for example,  $s_\alpha$  for  $\alpha = e_j - e_{j-1}$  or  $\alpha = e_{i+1} - e_i$  maps  $e_j - e_i$  to  $e_{j-1} - e_i$  or  $e_j - e_{i+1}$ , both of which have lower height. Repeating this, the process terminates at some  $e_k - e_{k-1}$ . Notice that the root of minimal height in  $W'\beta$  is not unique.

Let us write  $s_{j,i}$  for  $s_{e_j - e_i}$  and  $s_i = s_{i+1,i}$ . Then  $s_{i+1} \cdots s_{j-1}(e_j - e_i) = e_{i+1} - e_i$  and we obtain

$$s_{e_j - e_i} = s_{j-1} \cdots s_{i+1} s_i s_{i+1} \cdots s_{j-1}$$

For example, for  $e_3 - e_1$ , we get  $s_2(e_3 - e_1) = e_2 - e_1$  and  $s_{3,1} = s_2 s_1 s_2$ . ◇

**Exercise 3.14.** Let  $\Phi = \{e_i - e_j : i \neq j\}$  be the root system of type  $A_{n-1}$ . Every subsets  $U \subseteq \Phi$  determines a directed graph  $D = (V, A)$  on nodes  $V = \{1, \dots, n\}$  and arcs  $A = \{(i, j) : e_i - e_j \in U\}$ .

- (1) Show that  $U$  generates  $\mathfrak{S}_n$  if and only if  $D$  is connected.
- (2) Show that any minimal generating set  $U$  has the same cardinality.
- (3) How many minimal generating sets  $U$  are there?
- (4) How many simple systems are contained in  $\Phi$ ? ✎

**Exercise 3.15.** Exercise 3.14 suggests that  $U$  minimally generates  $\mathfrak{S}_n$  if and only if  $U$  is a basis for  $\{v \in V : v_1 + \cdots + v_n = 0\}$ . Show that  $\mathfrak{S}_n$  is special in that respect. ✎

#### 4. Reflection arrangements and the length function

Let us interpret what we did geometrically. Let  $W$  be a finite reflection group with an arbitrary but fixed root system  $\Phi$ . Recall that the reflection arrangement associated to  $W$  is the collection of hyperplanes  $\mathcal{A}(W) = \{H_\alpha : \alpha \in \Phi\}$ . A vector  $c \in V$  is generic if and only if  $c$  is not contained in any of the hyperplanes in  $\mathcal{A}(W)$ . Let  $\Phi_0^+$  be an arbitrary but fixed positive system. For  $c \in V \setminus 0$ , we write  $H_c^\geq := \{v : \langle c, v \rangle \geq 0\}$  for the **positive halfspace** that is bounded by the hyperplane  $H_c$  and  $H_c^> := H_c^\geq \setminus H_c$  for the **open halfspace**. Now  $c \in V \setminus 0$  yields  $\Phi_0^+$  if and only if  $\Phi_0^+ = \Phi \cap H_c^>$ , that is,  $\langle \alpha, c \rangle > 0$  for all  $\alpha \in \Phi_0^+$ . More generally, for  $v \notin \bigcup \mathcal{A}(W)$ , let  $\sigma_\alpha = \text{sgn}\langle \alpha, v \rangle \in \{-1, +1\}$  for all  $\alpha \in \Phi_0^+$ . Then the connected component of  $V \setminus \bigcup \mathcal{A}(W)$  containing  $v$  is

$$C_\sigma^\circ := \{v \in V : \langle \sigma_\alpha \alpha, v \rangle > 0 \text{ for all } \alpha \in \Phi_0^+\}$$

and yields the positive system  $\{\sigma_\alpha \alpha : \alpha \in \Phi_0^+\}$ . Thus, the connected components of

$$V \setminus \bigcup \mathcal{A}(W)$$

are in bijection to the positive (and hence simple) systems of  $W$ . We write  $C_\sigma$  for the topological closure of  $C_\sigma^\circ$ . Since  $C_\sigma^\circ$  is non-empty, we get

$$C_\sigma = \{v \in V : \langle \sigma_\alpha \alpha, v \rangle \geq 0 \text{ for all } \alpha \in \Phi_0^+\}.$$

The closures of the various connected components are called the **regions** or **chambers** of  $\mathcal{A}(W)$ , which we collect in  $\mathcal{R}(W)$ .

A set  $C \subseteq V$  is a **polyhedral cone** if there are  $\beta_1, \dots, \beta_m \in V$  with

$$C = \{v : \langle \beta_i, v \rangle \geq 0 \text{ for all } i = 1, \dots, m\} = H_{\beta_1}^\geq \cap H_{\beta_2}^\geq \cap \cdots \cap H_{\beta_m}^\geq.$$

The choice of  $\beta_i$  is not unique. Indeed, for  $\mu_1, \dots, \mu_m \geq 0$  define  $\beta = \mu_1 \beta_1 + \cdots + \mu_m \beta_m$ . Then  $\langle \beta, v \rangle \geq 0$  for all  $v \in C$  and  $C = C \cap H_\beta^\geq$ . We call the cone  $C$  **simplicial** if there is a choice of linearly independent  $\beta_1, \dots, \beta_m$ . It is easy to see that  $C$  is then linearly isomorphic to  $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^{n-m}$  if  $n = \dim V$ . We call an arrangement of hyperplanes **simplicial** if all regions are simplicial.

**Corollary 4.1.** *For any finite reflection group  $W$ , the arrangement  $\mathcal{A}(W)$  is simplicial.*

**PROOF.** Let  $\Phi^+$  be a positive system with associated chamber  $C = \{v : \langle \beta, v \rangle \geq 0 \text{ for all } \beta \in \Phi^+\}$ . Let  $\Delta = \{\alpha_1, \dots, \alpha_n\} \subseteq \Phi^+$  be the simple system. We claim that

$$C = \{v : \langle \alpha, v \rangle \geq 0 \text{ for all } \alpha \in \Delta\},$$

which then yields the result using Corollary 3.8. Let  $C'$  be the right-hand side. Since  $\Delta \subseteq \Phi^+$ , we get  $C \subseteq C'$ . Now for any  $\beta \in \Phi^+$ , there are  $b_1, \dots, b_n \geq 0$  such that  $\beta = b_1\alpha_1 + \dots + b_n\alpha_n$ . If  $c \in C'$ , then  $\langle \beta, c \rangle = b_1\langle \alpha_1, c \rangle + \dots + b_n\langle \alpha_n, c \rangle \geq 0$ , which shows  $c \in C$ .  $\square$

**Example 4.2** (Regions of types  $A_{n-1}$  and  $B_n$ ). We already did most of the leg work at the end of Section 1. The chambers of type  $A_{n-1}$  are the cones

$$\{v \in \mathbb{R}^n : v_{\tau(1)} < v_{\tau(2)} < \dots < v_{\tau(n)}\}$$

as  $\tau$  varies over all permutations  $\tau \in \mathfrak{S}_n$ . Each cone is linearly isomorphic to  $\mathbb{R}_{\geq 0}^{n-1} \times \mathbb{R}$ . In type  $B_n$ , the simple systems in Example 3.10 shows that the regions are of the form

$$\{v \in \mathbb{R}^n : 0 < \rho_1 v_{\tau(1)} < \rho_2 v_{\tau(2)} < \dots < \rho_n v_{\tau(n)}\}$$

as  $\tau$  varies over all permutations  $\tau \in \mathfrak{S}_n$  and  $\rho_1, \dots, \rho_n \in \{-1, +1\}^n$ .  $\diamond$

The group  $W$  acts on the set of regions  $\mathcal{R}(W)$ . For every  $w \in W$ ,  $w\Phi_0^+$  is also a positive system as  $w\Phi_0^+ = \Phi \cap H_{wc}^>$ . Likewise, if  $\Delta_0 \subseteq \Phi_0^+$  is the associated simple system, then  $w\Delta_0$  is a simple system for all  $w \in W$ . The next thing that we want to verify is that every positive (and hence simple) system is of the form  $w\Phi_0^+$  for some  $w \in W$ . Stronger even, we want to show that  $W$  acts **simply transitive** on the chambers of  $\mathcal{A}(W)$ , that is:

**THEOREM 4.3.** *Let  $\Phi^+$  be a positive system. Then there is a unique  $w \in W$  with  $\Phi^+ = w\Phi_0^+$ .*

Let  $C \in \mathcal{R}(W)$  be a chamber. A hyperplane  $H \in \mathcal{A}(W)$  is a **wall** of  $C$  if  $\text{span}(C \cap H) = H$ . If  $C = H_{\beta_1}^> \cap \dots \cap H_{\beta_m}^>$  is simplicial and  $\beta_1, \dots, \beta_m$  linearly independent, then  $H_{\beta_i}$  is a wall of  $C$  for each  $i = 1, \dots, m$ .

If  $H$  is a wall, then  $F = C \cap H$  is a **facet** (or **panel**) of  $C$ . For every facet  $F$  of  $C$  there is a unique chamber  $C' \in \mathcal{R}(W)$  with  $F = C \cap C'$ . We call  $C$  and  $C'$  **adjacent**. A hyperplane  $H \in \mathcal{A}(W)$  **separates** a chamber  $C$  from  $C'$  if  $C$  and  $C'$  lie on different sides of  $H$ . If  $C, C'$  are adjacent, then  $\text{span}(C \cap C')$  is the unique hyperplane that separates  $C$  from  $C'$ . We call  $C_1, C_2, \dots, C_k$  of distinct chambers a **gallery** if  $C_i, C_{i+1}$  are adjacent for  $1 \leq i < k$ .

**Proposition 4.4.** *Let  $C, C'$  be regions of the reflection arrangement  $\mathcal{A}(W)$ . If  $C, C'$  are adjacent with separating hyperplane  $H_\alpha \in \mathcal{A}(W)$ . Then  $s_\alpha(C) = C'$ .*

**PROOF.**  $W$  acts on  $\mathcal{A}(W)$  and hence on the regions  $\mathcal{R}(W)$ . In particular  $s_\alpha$  fixes the facet  $C \cap C' \subset H_\alpha$  pointwise. As  $s_\alpha$  exchanges  $H_\alpha^<$  and  $H_\alpha^>$ , this proves the claim.  $\square$

**PROOF OF THEOREM 4.3 – EXISTENCE.** Let  $C_0$  and  $C$  be the chambers corresponding to the positive systems  $\Phi_0^+$  and  $\Phi^+$ , respectively. Pick points  $p \in C_0^\circ, q \in C^\circ$  such that the segment  $[p, q] = \{p + \lambda(q - p) : 0 \leq \lambda \leq 1\}$  does not meet any of the finitely many codimension-2 subspace  $H \cap H'$  for  $H, H' \in \mathcal{A}(W)$  and  $H \neq H'$ . Let  $H_{\beta_1}, H_{\beta_2}, \dots, H_{\beta_k} \in \mathcal{A}(W)$  be the ordered sequence of hyperplanes that meet the segment  $[p, q]$  in distinct points. This means that there are chambers  $C_0, C_1, \dots, C_k = C$  that meet  $[p, q]$  such that  $C_i C_{i-1}$  are adjacent and separated by  $H_{\beta_i}$  for  $i = 1, \dots, k$ . It follows from Proposition 4.4 that  $C_i = s_{\beta_i}(C_{i-1})$  for  $i = 1, \dots, k$  and hence  $C = wC_0$  for  $w = s_{\beta_k} \dots s_{\beta_1}$ ; see Figure 3. On the level of positive systems, this means  $\Phi^+ = w\Phi_0^+$ .  $\square$

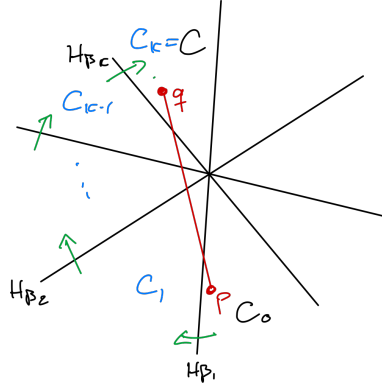


FIGURE 3. Illustration for proof of Theorem 4.3.

This proof paints a very geometric picture that we can also use to give an alternative proof of Theorem 3.12.

**GEOMETRIC PROOF OF THEOREM 3.12.** It suffices to show that every  $s_\beta$  for  $\beta \in \Phi^+$  can be written as a product of simple reflections. As before let  $C_0$  be the chamber corresponding to  $\Delta_0$  and pick points  $p \in C_0^\circ$  and  $q \in H_\beta$  such that the segment misses the finitely many codimension-2 intersections of hyperplanes in  $\mathcal{A}(W)$ . We again obtain a sequence of chambers  $C_0, C_1, \dots, C_k$  such that  $H_\beta$  is a wall for  $C_k$  (but for none of the other chambers). If  $k = 0$ , then  $H_\beta$  is a wall of  $C_0$  and hence  $\beta \in \Delta_0$  and  $s_\beta$  is a simple reflection.

If  $k > 0$ , then let  $H_{\beta_i}$  be the hyperplane separating  $C_{i-1}$  from  $C_i$  and set  $\beta_{k+1} = \beta$ . Moreover, let  $t_i = s_{\beta_i}$  be the reflection in  $H_{\beta_i}$ . Note that  $t_1$  is a simple reflection, that  $C_{i-1} = t_i(C_i)$  and hence  $t_1 \cdots t_{i-1}(H_{\beta_i})$  is a wall of  $C_0$ . It follows that  $s_i = t_1 t_2 \cdots t_{i-1} t_i t_{i-1} \cdots t_2 t_1$  is a simple reflection. By induction, we get that  $t_i = t_{i-1} \cdots t_2 t_1 s_i t_1 t_2 \cdots t_{i-1}$  is a product of simple reflections.  $\square$

Let us harvest this geometric perspective further!

**Construction 4.5.** If  $w = s_1 s_2 \cdots s_r$  for simple reflections  $s_i = s_{\alpha_i}$ , then define

$$\begin{aligned} t_1 &:= s_1 \\ t_2 &:= s_1 s_2 s_1 \\ &\vdots \\ t_r &:= s_1 s_2 \cdots s_{r-1} s_r s_{r-1} \cdots s_2 s_1. \end{aligned}$$

Then each  $t_i$  is a reflection in the hyperplane  $H_{\beta_i}$  with  $\beta_i = s_1 \cdots s_{i-1} \alpha_i$ . Moreover

$$t_k t_{k-1} \cdots t_1 = s_1 s_2 \cdots s_k$$

for all  $k = 1, \dots, r$ . Most importantly, if we define

$$C_k := t_k(C_{k-1}) = t_k t_{k-1} \cdots t_1 C_0 = s_1 s_2 \cdots s_k C_0$$

for  $k = 1, \dots, r$ , then  $C_0, \dots, C_r = wC$  is a gallery and  $t_i$  is the reflection in the unique hyperplane separating  $C_{i-1}$  and  $C_i$ .

As before let  $C_0$  be the chamber corresponding to  $\Phi_0^+$ . Note that  $C_0 \subseteq H_\beta^\geq$  for all  $\beta \in \Phi_0^+$ . For every chamber  $C$ , let us write  $\Phi_0^+(C) = \{\beta \in \Phi^+ : C \subseteq H_\beta^\leq\}$ . That is,  $\Phi_0^+(C)$  records the hyperplanes separating  $C$  from  $C_0$ . Moreover,

$$C = \bigcap_{\alpha \in \Phi_0^+(C)} H_\alpha^\leq \cap \bigcap_{\alpha \in \Phi_0^+ \setminus \Phi_0^+(C)} H_\alpha^\geq$$

If  $C$  corresponds to the positive system  $\Phi^+$ , then  $\Phi_0^+(C) = \Phi_0^+ \cap -\Phi^+$ . We write  $n(C) := |\Phi_0^+(C)|$  as the number of hyperplanes separating  $C$  from  $C_0$ .

**Exercise 4.6.** Let  $\mathcal{A}$  be a collection of finitely many hyperplanes in  $V$  and  $\mathcal{R} = \mathcal{R}(\mathcal{A})$  its set of regions. For any two regions  $C, C' \in \mathcal{R}$  write  $n(C, C')$  for the number of hyperplanes separating  $C$  from  $C'$ .

- (1) Show that  $n$  defines a metric on  $\mathcal{R}$ .
- (2) Let  $p \in C^\circ$  and  $q \in (C')^\circ$  be generic. Show that  $n(C, C')$  is the number of hyperplanes  $H \in \mathcal{A}$  that meet  $[p, q]$ .
- (3) Let  $C = C_0, C_1, \dots, C_k = C'$  be a gallery and  $H_i = \text{span}(C_{i-1} \cap C_i)$  the walls along the gallery. Show that if  $k > n(C, C')$ , then there are  $i < j$  such that  $H_i = H_j$ .  $\square$

Let  $C_0$  be the chamber for  $\Phi_0^+$  and  $w \in W$ . We define  $\Phi_0^+(w) := \Phi_0^+(wC_0)$  and  $n(w) := n(wC_0)$ .

**Lemma 4.7.** *Let  $w = s_1 \cdots s_r$  such that  $n(w) < r$ , then there are  $i < j$  such that*

$$w = s_1 \cdots \widehat{s}_i \cdots \widehat{s}_j \cdots s_r,$$

where  $\widehat{s}_i$  and  $\widehat{s}_j$  stands for omission.

PROOF. Consider the gallery  $C_0, \dots, C_r$  of Construction 4.5 for  $w = s_1 \cdots s_r$ . If  $r > n(w)$ , then the gallery traverses a hyperplane twice by Exercise 4.6. That is,  $t_i = t_j$  for some  $i < j$  but this means

$$s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1 = s_1 \cdots s_{i-1} s_i s_{i+1} \cdots s_{j-1} s_j s_{j-1} \cdots s_{i+1} s_i s_{i-1} \cdots s_1,$$

which reduces to

$$s_1 \cdots s_{i-1} = s_1 \cdots s_{i-1} s_i s_{i+1} \cdots s_{j-1} s_j s_{j-1} \cdots s_{i+1},$$

which reduces to

$$s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} = s_1 \cdots s_{i-1} s_i s_{i+1} \cdots s_{j-1} s_j. \quad \square$$

By Theorem 3.12 for every  $w \in W$  there are  $\alpha_1, \dots, \alpha_k \in \Delta_0$  such that  $w = s_{\alpha_1} \cdots s_{\alpha_k}$ . If  $k$  is minimal, then we call  $s_{\alpha_1} \cdots s_{\alpha_k}$  a **reduced expression** for  $w$  and define the **length** of  $w$  (relative to  $\Delta_0$ ) as  $\ell(w) := k$ .

**Proposition 4.8.** *Let  $w \in W$ .*

- (i)  $\ell(e) = 0$  and  $\ell(w) = 1$  if and only if  $w = s_\alpha$  for some  $\alpha \in \Delta_0$ .
- (ii)  $\ell(w) = \ell(w^{-1})$ .
- (iii)  $\ell(w w') \leq \ell(w) + \ell(w')$ .
- (iv)  $\ell(s_\alpha w) = \ell(w s_\alpha) = \ell(w) \pm 1$  for all  $\alpha \in \Delta_0$ .

PROOF. (i) is clear. For (ii): If  $w = s_{\alpha_1} \cdots s_{\alpha_k}$ , then  $w^{-1} = s_{\alpha_k} \cdots s_{\alpha_1}$ , which shows  $\ell(w^{-1}) \leq \ell(w)$ . Since  $w = (w^{-1})^{-1}$ , we also get  $\ell(w) \leq \ell(w^{-1})$ . For (iii) just compose reduced expressions for  $w$  and  $w'$ .

For (iv), first note that  $|\ell(s_\alpha w) - \ell(w)| \leq 1$ . Now  $\det(s_\alpha) = -1$  (as  $H_\alpha$  is the eigenspace to the eigenvalue 1 and  $(H_\alpha)^\perp = \mathbb{R}\alpha$  is the eigenspace to eigenvalue  $-1$ ). Thus  $\det(w) = (-1)^{\ell(w)}$ . This implies  $\ell(s_\alpha w) \neq \ell(w)$ .  $\square$

The next theorem gives a geometric interpretation of the length function.

**THEOREM 4.9.** *Let  $W$  be a finite reflection group with simple positive systems  $\Delta_0 \subseteq \Phi_0^+$ . For all  $w \in W$ , we have  $\ell(w) = n(w) = |\Phi_0^+ \cap w^{-1}(-\Phi_0^+)|$ .*

PROOF. Let  $w = s_1 s_2 \cdots s_r$  be a reduced expression with simple reflections  $s_1, \dots, s_r$ . Then there are at most  $r$  distinct hyperplanes in the gallery of Construction 4.5 that separate  $C_0$  from  $C_r = wC_0$ . By Exercise 4.6, we obtain  $n(w) = n(wC_0) \leq r = \ell(w)$ . However, if  $n(wC_0) < r$ , then Lemma 4.7 yields  $w = s_1 \cdots \widehat{s}_i \cdots \widehat{s}_j \cdots s_r$  which contradicts the assumption that the expression was reduced.  $\square$

PROOF OF THEOREM 4.3 – UNIQUENESS. Assume that there is  $w \in W$  such that  $w\Phi_0^+ = \Phi_0^+$ . But this means that  $n(w) = 0$  and by Theorem 4.9 we get  $w = e$ .  $\square$

The following deletion condition follows directly from Theorem 4.9 and Lemma 4.7.

**Corollary 4.10.** *Let  $w \in W$  and  $s_\alpha$  a simple reflection.*

- (1)  $\ell(ws_\alpha) = \ell(w) + 1$  if and only if  $w\alpha \in \Phi_0^+$ .
- (2)  $\ell(ws_\alpha) = \ell(w) - 1$  if and only if  $w\alpha \in -\Phi_0^+$ .
- (3) (Deletion Condition) *Let  $w = s_1 \cdots s_r$  be a product of simple reflections. If the expression is not reduced, then there exist indices  $1 \leq i < j \leq r$  with  $w = s_1 \cdots \widehat{s}_i \cdots \widehat{s}_j \cdots s_r$ .*
- (4) (Exchange Condition) *Let  $w = s_1 \cdots s_r$  be a product of simple reflections and  $s$  some simple reflection. If  $\ell(ws) < \ell(w)$ , then there is  $1 \leq i \leq r$  with  $w = s_1 \cdots \widehat{s}_i \cdots s_r s$ . In particular,  $w$  has a reduced expression ending in  $s$  if and only if  $\ell(ws) < \ell(w)$ .*

**Exercise 4.11.** Proof Corollary 4.10. ✍

**Exercise 4.12.** Let  $W$  be a finite reflection group with simple system  $\Delta$  and length function  $\ell$  relative to  $\Delta$ .

- (1) Show that there is a unique element  $w_0 \in W$  of maximal length. This is called the **longest element** of  $W$  relative to  $\Delta$ . What is the length?
- (2) Show that  $w_0$  is an involution, that is  $w_0^2 = e$ .
- (3) Prove that in every reduced expression of  $w_0$ , every simple reflection must occur at least once.
- (4) Let  $w$  with reduced expression  $w = s_1 \cdots s_r$ . Show that there is a  $w'$  with reduced expression  $w' = s_{r+1} \cdots s_m$  such that  $s_1 \cdots s_r s_{r+1} \cdots s_m$  is a reduced expression for  $w_0$ .
- (5) Show that for every  $w \in W$  there is a simple system  $\Delta$  such that  $w$  is the longest element. ✍

## 5. Fundamental domains and parabolic subgroups

For a group  $G$  acting on a space  $V$ , a **fundamental domain** is a *nice* (e.g. connected) subset  $D \subset V$  such that  $D$  meets every orbit  $Gv$  for  $v \in V$  in a unique point. Fundamental domains for arbitrary group actions might be tricky. For example, if  $G$  acts on  $V = \mathbb{R}^2$  by rotation of  $\frac{\pi}{2}$ , then a fundamental domain is given by

$$D = \{(0,0)\} \cup \{(x,0) : x > 0\} \cup \{(x,y) : x,y > 0\}$$

see Figure 4. This set is neither open nor closed and it can be shown(?) that there is no nice connected fundamental domain for this action. The situation for finite reflection groups is rather different. If, for example,  $W = \{e, s_\alpha\}$ , then  $H_\alpha^{\geq}$  or  $H_\alpha^{\leq}$  acts as a fundamental domain, which is as nice as possible (closed, convex).

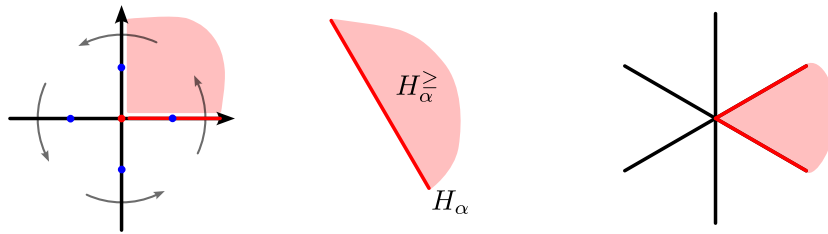


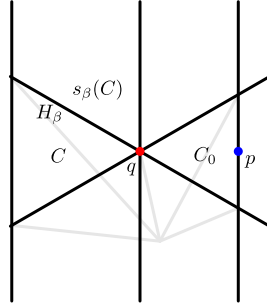
FIGURE 4. Fundamental domain for a rotation group acting on the plane. The blue points are in one orbit. The reddish point, half line and positive orthant are a fundamental domain.

**THEOREM 5.1.** *Let  $W$  be a finite reflection group. Then any  $C \in \mathcal{R}(W)$  is a fundamental domain.*

**PROOF.** Let  $C_0 \in \mathcal{R}(W)$  be a chamber with simple system  $\Delta_0$ . For  $v \in V$  we need to show that  $Wv$  meets  $C_0$  in exactly one point. To see that there is  $w \in W$  with  $wv \in C_0$ , let  $C \in \mathcal{R}(W)$  with  $v \in C$ . By Theorem 4.3, there is a unique  $w \in W$  with  $C = wC_0$ , that is,  $C_0 = w^{-1}C$  and  $w^{-1}v =: p \in C_0$ .



Assume that there is  $w' \in W$  with  $w'v \in C_0$ . This means that  $p := w'v \in C_0$  and  $wp =: q \in C_0$ . If  $p$  is in the interior of  $C_0$ , then so is  $q$  and  $wC_0 = C_0$  but this can only happen for  $w = e$ . Thus,  $p, q \in \partial C_0$ . We wish to prove that  $p = q$ .



Let  $C = wC_0$ . We prove the claim by induction on  $\ell(w)$ , the number of hyperplanes separating  $C_0$  from  $C$ . Observe that  $C_0 \cap C \neq \emptyset$  as both contain  $q$ . We claim that there is a wall  $H_\beta$  of  $C$  that separates  $C_0$  from  $C$  and  $q \in C \cap H_\beta$ . If not, then every wall  $H$  of  $C$  that contains  $q$  satisfies  $C_0 \subseteq H^\geq$  but this is absurd!. Now  $C' = s_\beta(C)$  also contains  $q$  and hence  $s_\beta wp = q$  and  $\ell(s_\beta w) < \ell(w)$ . The result now follows by induction.  $\square$

As an upshot of the proof, we can determine the **isotropy group** (or **stabilizer group**)  $W_v := \{w \in W : wv = v\}$  for every point in  $V$ . For  $v \in C_0$ , the induction actually proves that if we let  $J := \{\alpha \in \Delta_0 : v \in H_{\alpha_0}\}$ , then  $W_v$  is generated by the reflections  $\{s_\alpha : \alpha \in J\}$ . Conversely for any  $J \subseteq \Delta$ , the subgroup  $W_J = \langle s_\alpha : \alpha \in J \rangle$  is the isotropy group for all points in the linear subspace  $L_J := \bigcap_{\alpha \in J} H_\alpha$ . We call  $W_J$  a **standard parabolic subgroup** relative to the simple system  $\Delta_0$ . For any  $w \in W$ , we call  $wW_Jw^{-1}$  a **parabolic subgroup**. That is,  $W'$  is a parabolic subgroup if  $W'$  is a standard parabolic subgroup for *some* simple system  $\Delta$ .

Since for any  $v \in V$ , there is  $w \in W$  with  $wv \in C_0$ , we deduce.

**Corollary 5.2.** *Every isotropy group is a parabolic subgroup.*

**Exercise 5.3.** Find an example of a reflection group  $W$  and a subgroup  $W' \subset W$  that is generated by reflections but that is not a parabolic subgroup.

We can make use of parabolic subgroups to study the **length generating polynomial**

$$W(q) = \sum_{w \in W} q^{\ell(w)}.$$

**Example 5.4** (Dihedral groups). For the dihedral group  $\mathcal{D}_m$ , we can compute by inspection

$$\mathcal{D}_m(q) = 1 + 2q + 2q^2 + \dots + 2q^{m-1} + q^m$$

Let us define  $[n]_q := 1 + q + q^2 + \dots + q^{n-1}$ , the  **$q$ -integer**. Then

$$\mathcal{D}_m(q) = (1 + q)(1 + q + q^2 + \dots + q^{m-1}) = [2]_q [m]_q. \quad \diamond$$

To be able to compute  $W(q)$ , we want to use that for  $W_I$ , we can decompose  $W$  into cosets  $wW_I$ . If we can find suitable coset representative on which the length function is additive, then this might pave the way for an inductive procedure.

We defined  $W_I$  as the reflection group defined by a subset of simple reflections. It follows from Exercise 3.1 that  $\Phi_I = \Phi \cap \text{span}(I)$  is a root system for  $W_I$  with simple system  $I$ . Relative to  $I$ , we can define a length function  $\ell_I$  on  $W_I$ .

**Proposition 5.5.** *We have  $\ell_I(w) = \ell(w)$  for any  $w \in W_I$ .*

PROOF. We recall that  $\ell(w) = |\{\beta \in \Phi^+ : w\beta \in -\Phi^+\}|$ . For  $\beta \in \Phi^+ \setminus \Phi_I^+$  and  $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$  with  $c_\alpha \geq 0$ , there is some  $\gamma \in \Delta \setminus I$  with  $c_\gamma > 0$ . Thus, for all  $\alpha \in I$ , the coefficient of  $\gamma$  of  $s_\alpha(\beta)$  is still positive. Thus, for  $w \in W_I$  and  $\beta \in \Phi^+$ , this means  $w\beta \in -\Phi^+$  if and only if  $\beta \in \Phi_I^+$ . Hence  $\ell_I(w) = \ell(w)$ .  $\square$

For  $I \subseteq \Delta$  define

$$W^I := \{w \in W : \ell(ws_\alpha) > \ell(w) \text{ for all } \alpha \in I\}.$$

**Lemma 5.6.** *For every  $w \in W$  there is a unique  $u \in W^I$  and  $v \in W_I$  such that  $w = uv$  and  $\ell(w) = \ell(u) + \ell(v)$ .*

PROOF. For  $w \in W$ , let  $u \in wW_I$  with  $\ell(u)$  minimal and write  $uv = w$  for  $v \in W_I$ . Let  $u = s_1 \cdots s_k$  and  $v = s_{k+1} \cdots s_{k+l}$  be reduced expressions. If  $uv = s_1 \cdots s_k s_{k+1} \cdots s_{k+l}$  is not reduced, then by the Deletion Condition of Corollary 4.10 there are  $1 \leq i < j \leq k+l$  such that  $s_i$  and  $s_j$  can be removed without altering  $uv$ . Now  $j \leq k$  or  $k < i$  would contradict the reducedness of the expressions for  $u$  and  $v$ . Thus  $i < k < j$ , but then this means that  $u' = s_1 \cdots \widehat{s_i} \cdots s_k \in wW_I$  is of shorter length. Hence  $\ell(w) = \ell(uv) = \ell(u) + \ell(v)$ . If there is another  $u' \in wW_I = uW_I$  with  $\ell(u') = \ell(u)$ , then  $u = u'v'$  for some  $v' \in W_I$  and the same argument shows  $\ell(u) = \ell(u') + \ell(v')$ . But then  $\ell(v') = 0$  and  $v' = e$ .

Since  $u$  is chosen of minimal length in  $wW_I$ , this implies  $\ell(us) > \ell(u)$  for all  $s \in I$  and hence  $u \in W^I$ .  $\square$

The elements in  $W^I$  are called **minimal coset representatives**. For  $X \subseteq W$  we define  $X(q) := \sum_{w \in X} q^{\ell(w)}$ . Then Lemma 5.6 implies

**Corollary 5.7.** *For any  $J \subseteq \Delta$*

$$W(q) = W^J(q)W_J(q).$$

Let us interpret  $W^I$  geometrically. Recall that  $\ell(w^{-1}) = \ell(w)$  and  $(ws_\alpha)^{-1} = s_\alpha w^{-1}$ . Hence, we can write

$$W^I := \{w^{-1} : \ell(s_\alpha w) > \ell(w)\}.$$

The benefit of this small change is that according to Construction 4.5,  $\ell(s_\alpha w) > \ell(w)$  if and only if  $wC_0$  is *not* separated from  $C_0$  by the hyperplane  $H_\alpha$ . Hence  $W^I$  is in bijection to all chambers  $C \in \mathcal{R}(W)$  with  $C \subseteq \bigcap_{\alpha \in I} H_\alpha^>$ . Let  $\mathcal{R}(W)^I \subseteq \mathcal{R}(W)$  be the collection of all these chambers. The reflection arrangement  $\mathcal{A}(W_I)$  is a subarrangement of  $\mathcal{A}(W)$ . Every chamber  $C' \in \mathcal{R}(W_I)$  is the union of chambers of  $C$  of  $W$  with  $C \subseteq C'$ . The distinguished region  $C_0 \in \mathcal{R}(W)$  determines a distinguished region  $C'_0 \in \mathcal{R}(W_I)$  and  $\mathcal{R}(W)^I$  are precisely the regions of  $W$  that make up  $C'_0$ . Figure 5 illustrates the situation for  $\mathfrak{S}_4$ .

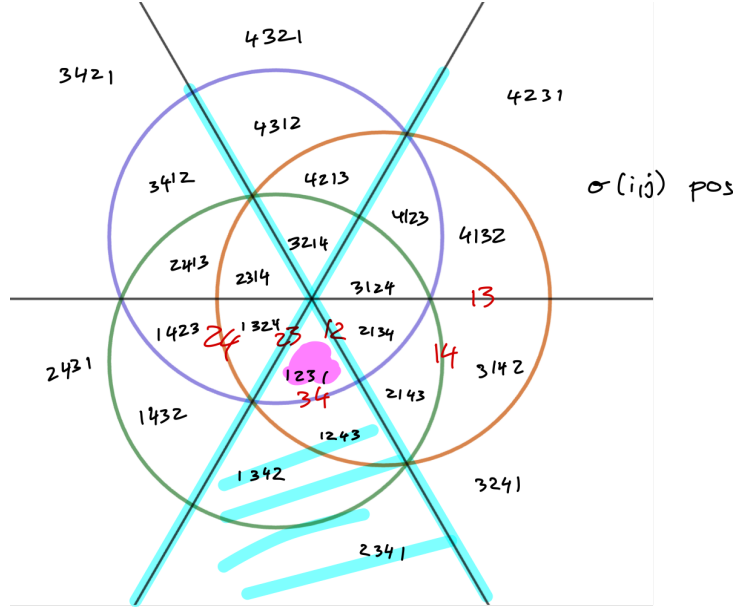


FIGURE 5. Stereographic projection of reflection arrangement of  $\mathfrak{S}_4$ . Permutations in black. The six reflections in red. Fundamental region in marker pink. Simple reflections for  $W_I$  in cyan. Black lines are reflection arrangement for  $W_I$ .

Whereas  $W_J$  is again a reflection group and we can hope to compute  $W_J(q)$  inductively, the set  $W^J$  is not a group. We can, however, still compute  $W^J(q)$  by inclusion-exclusion. For  $w \in W$  define

$$D_R(w) := \{\alpha \in \Delta : \ell(ws_\alpha) < \ell(w)\}. \quad (2)$$

We call  $\alpha \in D_R(w)$  a **(right) descent** of  $w$ . By the Exchange Condition of Corollary 4.10,  $D_R(w)$  are precisely those  $s_\alpha$  that occur as the final simple reflection in some reduced expression of  $w$ .

**Example 5.8** (Type  $A_{n-1}$ ). Recall that for  $\sigma \in \mathfrak{S}_n$ ,  $\ell(\sigma)$  is the number of inversions. For a simple reflection  $s_i = (i, i + 1)$ ,  $\sigma \circ (i, i + 1)$  exchanges  $\sigma(i)$  and  $\sigma(i + 1)$  in the one-line notation. Thus, if  $\ell(\sigma \circ (i, i + 1)) < \ell(\sigma)$ , then this means that  $\sigma(i) > \sigma(i + 1)$ , that is,  $i$  is a *descent* as commonly defined for permutations.  $\diamond$

For  $I \subseteq J \subseteq \Delta$ , also define

$$\begin{aligned} D_I^J &:= \{w \in W : I \subseteq D_R(w) \subseteq J\} \\ D_I &:= D_I^I = \{w \in W : D_R(w) = I\} \\ W^J &= D_\emptyset^{\Delta \setminus J}. \end{aligned}$$

Note that the last line is *not* a definition but a straightforward consequence of the definition of  $W^J$ . Note also that

$$D_I^J = \bigsqcup_{I \subseteq K \subseteq J} D_K \quad \text{and thus} \quad D_I^J(q) = \sum_{I \subseteq K \subseteq J} D_K(q).$$

**Proposition 5.9.** For  $I \subseteq J \subseteq \Delta$

$$D_I^J(q) = \sum_{J \setminus I \subseteq K \subseteq J} (-1)^{|J \setminus K|} W^{\Delta \setminus K}(q).$$

PROOF. Note that for any  $K \subseteq J$ , we can combine above insights to get

$$W^{\Delta \setminus K}(q) = D_\emptyset^K(q) = \sum_{L \subseteq K} D_L(q).$$

Thus

$$\sum_{J \setminus I \subseteq K \subseteq J} (-1)^{|J \setminus K|} W^{\Delta \setminus K}(q) = \sum_{L \subseteq J} D_L(q) \sum_{(J \setminus I) \cup L \subseteq K \subseteq J} (-1)^{|J \setminus K|}.$$

Using the binomial theorem, we get that the last sum is 1 if and only if  $(J \setminus I) \cup L = J$ , that is,  $I \subseteq L \subseteq J$ . Hence

$$\sum_{J \setminus I \subseteq K \subseteq J} (-1)^{|J \setminus K|} W^{\Delta \setminus K}(q) = \sum_{I \subseteq L \subseteq J} D_L(q) = D_I^J(q). \quad \square$$

For the next result we need the fact that relative to  $\Delta$ , there is a unique element  $w_0 \in W$  with  $\ell(w_0) = |\mathcal{A}(W)|$ . This is the **longest element** of Exercise 4.12.

**Corollary 5.10.**

$$\sum_{K \subseteq \Delta} \frac{(-1)^{|K|}}{W_K(q)} = \frac{q^{|\mathcal{A}|}}{W(q)}.$$

In particular for all  $I \subseteq \Delta$

$$\sum_{K \subseteq \Delta} (-1)^{|K|} \frac{|W_I|}{|W_K|} = \frac{1}{|W^I|}.$$

PROOF. Take  $I = J = \Delta$ . Then Proposition 5.9 and Corollary 5.7 yields

$$q^{|\mathcal{A}|} = D_{\Delta}^{\Delta}(q) = \sum_{K \subseteq \Delta} (-1)^{|\Delta \setminus K|} W^{\Delta \setminus K}(q) = \sum_{K \subseteq \Delta} (-1)^{|K|} W^K(q) = \sum_{K \subseteq \Delta} (-1)^{|K|} \frac{W(q)}{W_K(q)}. \quad \square$$

For the symmetric group something amazing happens. If we compute the length generating function, we get

$$\mathfrak{S}_n(q) = [n]_q [n-1]_q \cdots [2]_q [1]_q =: [n]_q!. \quad (3)$$

**Exercise 5.11.** For a permutation  $\tau \in \mathfrak{S}_n$  define  $I(\tau) = (a_1, \dots, a_n)$  by  $a_i := \#\{j > i : \tau(i) > \tau(j)\}$ . This defines a map  $I : \mathfrak{S}_n \rightarrow [n-1] \times [n-2] \times \cdots \times [2] \times [1]$ .

- (1) Show that  $I$  is injective. Hint: If  $I(\sigma) = I(\tau)$ , consider the maximal  $t$  with  $\sigma^{-1}(t) \neq \tau^{-1}(t)$ .
- (2) Conclude that  $I$  is bijective.
- (3) Prove (3) using  $I$ . ✍

This factorization into polynomials all whose coefficients are 1 happens for *all* finite reflection groups.

**THEOREM 5.12.** For any finite reflection group  $W$  there are positive integers  $e_1, e_2, \dots, e_k$  such that

$$W(q) = [e_1]_q [e_2]_q \cdots [e_k]_q.$$

## 6. The Coxeter complex

The reflection arrangement  $\mathcal{A}(W)$  (and any arrangement, actually) induces a decomposition of the real vector space  $V$ , that we want to study now. Recall that a polyhedral cone is a set of the form  $C = \{v \in V : \langle \beta_i, v \rangle \geq 0 \text{ for } i = 1, \dots, m\}$ . A linear hyperplane  $H_{\alpha} \subseteq V$  is **valid** for  $C$  if  $C \subseteq H_{\alpha}^{\geq}$  and a **face** of  $C$  is a subset  $F = C \cap H_{\alpha}$ , where  $H_{\alpha}$  is a valid hyperplane. Note that every face of a cone is again a cone. We also let  $F = C$  be a face. The dimension of a cone is  $\dim C = \dim \text{span}(C)$ .

For every  $J \subseteq [m]$ , the set

$$C_J := \{v \in C : \langle \beta_i, v \rangle = 0 \text{ for } i \in J\}$$

is a face of  $C$  with respect to the valid hyperplane  $H_{\beta}$  with  $\beta = \sum_{i \in J} \beta_i$ . Conversely, it can be shown that every face is of the form  $C_J$  for some  $J$ .

If  $C$  is a *simplicial* cone, that is, if  $\beta_1, \dots, \beta_m$  are linearly independent, then  $C_J = C_{J'}$  if and only if  $J = J'$ . Moreover,  $\dim C_J = \dim V - |J|$ . That is, the **codimension** of  $C_J$  is  $\text{codim } C_J = \dim V - \dim C_J = |J|$ .

This gives us an easy way to count faces of a simplicial cone. The number of faces of codimension  $k$  is the coefficient of  $z^k$  of the polynomial

$$f_C(z) := \sum_{J \subseteq [m]} z^{|J|} = \sum_{l=0}^m \binom{m}{l} z^l = (1+z)^m.$$

We'll see next why it is advantageous to use the codimension instead of dimension.

For our reflection arrangements  $\mathcal{A} = \mathcal{A}(W)$ , the space  $V$  is decomposed into cones of various dimensions. Let  $\Phi_0^+$  be a fixed positive system. For a point  $p \in V$ , define  $\kappa_\alpha(p) := \text{sgn}\langle \alpha, p \rangle \in \{-1, 0, +1\}$  for each  $\alpha \in \Phi_0^+$ . Then

$$C_\kappa := \left\{ v \in V : \begin{array}{l} \langle \kappa_\alpha \alpha, v \rangle \geq 0 \text{ for } \kappa_\alpha \neq 0 \\ \langle \alpha, v \rangle = 0 \text{ for } \kappa_\alpha = 0 \end{array} \right\}$$

is a non-empty cone that contains  $p$  in its relative interior<sup>1</sup>; see Figure 6 for an example.

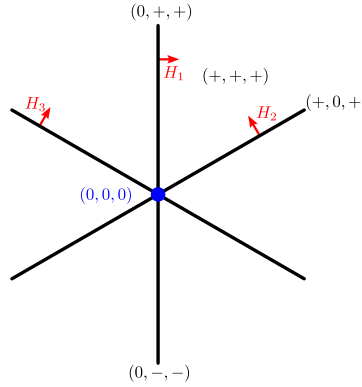


FIGURE 6. Decomposition of  $\mathbb{R}^2$  into faces of various dimensions by  $\mathcal{A}$ . The  $f$ -polynomial is  $f_{\mathcal{A}}(z) = 6 + 6z + z^2$ .

Since all chambers of  $\mathcal{A}$  are simplicial, the codimension of  $C_\kappa$  is  $d(\kappa) := |\{\alpha \in \Phi^+ : \kappa_\alpha = 0\}|$ . We define the **face vector** or  **$f$ -vector**  $f(\mathcal{A})$  so that  $f_k(\mathcal{A})$  is the number of cones  $C_\kappa$  of dimension  $k$ , for  $k = 0, 1, \dots, n = \dim V$ . We also define the  **$f$ -polynomial** as

$$f_{\mathcal{A}}(q) := f_n(\mathcal{A}) + f_{n-1}(\mathcal{A})q + \dots + f_0(\mathcal{A})q^n.$$

Now, if  $W$  is not essential, that is if  $V^W = \{v \in V : wv = v \text{ for all } w \in W\} \neq 0$ . Then we can restrict the action of  $W$  to  $U := (V^W)^\perp$ . Thus  $W$  is an essential reflection group in  $U$  with reflection arrangement  $\mathcal{A}|_U$ . If  $F$  is a face of  $\mathcal{A}$ , then  $F \cap U$  is a face of  $\mathcal{A}|_U$ . The faces change but the codimension is left untouched.

For a reflection arrangement  $\mathcal{A} = \mathcal{A}(W)$ , we can interpret  $f_k(\mathcal{A})$  algebraically. Again, let  $\Delta \subseteq \Phi^+ \subset \Phi$  be fixed and let  $C$  be the fundamental domain with respect to  $\Delta$ . Let  $W_J$  be a standard parabolic subgroup generated by  $s_\alpha$  for  $\alpha \in J \subseteq \Delta$ . Recall that the rank of  $W_J$  is the codimension of the fixed space  $V^{W_J} = L_J = \bigcap_{\alpha \in J} H_\alpha$ , which is precisely  $|J|$ . Moreover  $J$  is a simple system for  $W_J$ , which we can identify with the face

$$C^J := \{v : \langle \alpha, v \rangle = 0 \text{ for all } \alpha \in \Delta \setminus J, \langle \alpha, v \rangle \geq 0 \text{ for all } \alpha \in J\},$$

which has codimension  $|\Delta \setminus J|$ . Note that  $wJ$  is a simple system for the parabolic subgroup  $wW_J w^{-1}$  of the same rank and  $wJ$  is identified with  $wC^J$ . Let  $\Phi_J^+ = \Phi^+ \cap \text{cone}(J)$ , which is a positive system for  $W_J$ . Then

$$C^J = \bigcap_{\alpha \in \text{Pos}_J} H_\alpha^\geq \cap \bigcap_{\alpha \in \Phi^+ \setminus \text{Pos}_J} H_\alpha.$$

<sup>1</sup>That is, the interior of  $C_\kappa$  relative to the linear subspace  $\bigcap_{\kappa_\alpha=0} H_\alpha$ .

Hence  $wC^J = C^J$  if and only if  $w \in W_{\Delta \setminus J}$  and hence  $wC^J$  for  $w \in W^{\Delta \setminus J}$  represent the distinct simple systems for the parabolic subgroups that are conjugate to  $W_J$ , that is,  $wW_Jw^{-1}$  for  $w \in W$ . Therefore  $(f_{\mathcal{A}(W)})_k$  is the number of simple systems of parabolic subgroups of rank  $k$ . For example for  $k = n$ , we have  $J = \Delta$  and  $W_J = W$ . The number of simple systems is the number of regions of  $\mathcal{A}(W)$ , which is  $W = W^\emptyset$ . At the other extreme  $k = 0$ , we have  $J = \emptyset$  and  $W_J = \{e\}$ . This group has only one simple system represented by  $\bigcap \mathcal{A}(W)$  and  $W^\Delta = \{e\}$ . In summary

$$f_{\mathcal{A}(W)}(q) = \sum_{J \subseteq \Delta} |W^{\Delta \setminus J}| q^{|J|}.$$

**Example 6.1** (Dihedral groups). Let  $\mathcal{D}_m$  be the dihedral symmetry group of an  $m$ -gon. The arrangement  $\mathcal{A}_m = \mathcal{A}(\mathcal{D}_m)$  has  $m$  lines. The  $f$ -vector is then  $f(\mathcal{A}_m) = (1, 2m, 2m)$ .  $\diamond$

**Example 6.2** (Type  $A_{n-1}$ ). For every  $p \in \mathbb{R}^n$ , there is a unique ordered set partition  $M = (M_1, \dots, M_k)$  with  $M_1 \cup \dots \cup M_k = [n]$ ,  $M_r \neq \emptyset$  and  $M_r \cap M_s = \emptyset$  for all  $r < s$ . Such that

- (1)  $p_i = p_j$  if and only if  $i, j \in M_l$  for some  $l$ ;
- (2)  $p_i < p_j$  if and only if  $i \in M_r, j \in M_s$  with  $r < s$ .

The cone containing  $p$  is precisely the set of  $v \in V$  that give rise to the same set partition. The dimension of the cone is  $k$ , the length of the partition. The number of such *unordered* set partitions of  $n$  with  $k$  parts is counted by the famous **Stirling numbers of the second kind**  $S(n, k)$ . They are given by  $S(0, 0) = 1$ ,  $S(n, k) = 0$  for  $k > n$  or  $k = 0$  and

$$S(n, k) = kS(n-1, k) + S(n-1, k-1).$$

Here is a table

$S(n, k)$	$k = 1$	$2$	$3$	$4$	$5$	$6$
$n = 1$	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

It follows that  $f_k(\mathcal{A}) = k!S(n, k)$ , as we have  $k!$  ways to arrange the  $k$  parts of the partition. For example, for  $n = 3$ , we get  $f(\mathcal{A}(\mathfrak{S}_3)) = (0, 1, 2! \cdot 3, 3! \cdot 1) = (0, 1, 6, 6)$ , which is the  $f$ -vector for  $\mathcal{D}_3$  but the triangle is embedded in 3-space. For  $n = 4$ , we get

$$f(\mathcal{A}(\mathfrak{S}_3)) = (0, 1, 14, 36, 24). \quad \diamond$$

**Exercise 6.3.** Show the following identities of formal power series

- (1) For fixed  $k \geq 0$

$$\sum_{n \geq k} S(n, k) x^n = \frac{x^k}{(1-x)(1-2x) \cdots (1-kx)}.$$

- (2) For  $k \geq 0$

$$\sum_{n \geq k} S(n, k) \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k,$$

where  $e^x = \sum_{h \geq 0} \frac{x^h}{h!}$ .  $\textcircled{e}$

We will compute the  $f$ -vector or rather the  $f$ -polynomial of a simplicial arrangement  $\mathcal{A}$  by a technique called **half-open** decomposition. Let  $C_0$  be an arbitrary but fixed region. For any region  $C \in \mathcal{R}(\mathcal{A})$  let  $S(C)$  be the collection of hyperplanes  $H \in \mathcal{A}$  that separate  $C$  from  $C_0$ . We define

$$\widehat{C} := C \setminus \bigcup_{H \in S(C)} H.$$

We call  $\widehat{C}$  a **half-open cone**. Note that  $\widehat{C}_0 = C_0$  and that the half-open cone of  $-C_0$  is the interior of  $-C_0$ .

**Lemma 6.4.** *Let  $\mathcal{A}$  be a hyperplane arrangement with regions  $\mathcal{R}(\mathcal{A})$ . Then*

$$V = \bigsqcup_{C \in \mathcal{R}(\mathcal{A})} \widehat{C}.$$

PROOF. Let  $p \in C_0^\circ$  be arbitrary but fixed. Let  $v \in V$  be an arbitrary point. If  $v \in C^\circ$  for some region  $C$ , then  $\widehat{C}$  is unique region containing  $v$ . Otherwise, consider  $q := v - \varepsilon(v - p)$  for  $\varepsilon > 0$ . For  $\varepsilon > 0$  sufficiently small,  $q$  is contained in the interior of the unique  $C \in \mathcal{R}(\mathcal{A})$  for which  $v \in \partial C$ . Let  $H_\alpha \in \mathcal{A}$  be a hyperplane containing  $v$  and assume that  $C_0 \subset H_\alpha^\geq$ . Then  $\langle \alpha, v \rangle = 0$  and, by construction,  $\langle \alpha, q \rangle = \varepsilon \langle \alpha, p \rangle > 0$ , since  $p \in C_0^\circ$ . Thus  $C \subseteq H_\alpha^\geq$ . This shows that  $v \in \widehat{C}$ .

Assume that  $C'$  is another region with  $v \in \partial C'$ . Then there is a wall  $H$  of  $C'$  separating  $C'$  from  $C$ . By construction of  $C$ , we have that  $q$  and  $p$  lie on the same side, that is,  $H$  separates  $C'$  from  $C_0$  and hence  $v \notin \widehat{C}'$ .  $\square$

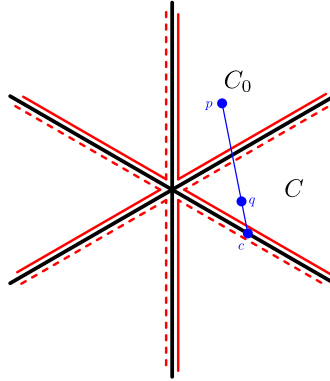


FIGURE 7. Half-open decomposition.

Let  $C = \{v : \langle a_i, v \rangle \geq 0 \text{ for } i = 1, \dots, n\}$  such that all  $H_{\alpha_i}$  are walls of  $C$ . Let  $D = D(C) := \{i : C_0 \subseteq H_{\alpha_i}^\leq\}$ . Then

$$\widehat{C} = \{v : \langle a_i, v \rangle \geq 0 \text{ for } i \notin D \text{ and } \langle a_i, v \rangle > 0 \text{ for } i \in D\}.$$

The faces of  $\widehat{C}$  are precisely those faces  $F$  of  $C$  that do not lie in any of the hyperplanes  $H_{\alpha_i}$  for  $i \in D$ . If  $C$  is simplicial, then they are easy to count:

$$f_{\widehat{C}}(z) = \sum_{J \subseteq [m] \setminus D} z^{|J|} = (1 - z)^{m - |D|}.$$

Note that if  $0 \leq |D(C)| \leq m$  and we define the  **$h$ -polynomial**

$$h_{\mathcal{A}}(z) = \sum_{C \in \mathcal{R}(\mathcal{A})} z^{m - |D(C)|} = h_m(\mathcal{A}) + h_{m-1}(\mathcal{A})z + \dots + h_0(\mathcal{A})z^m.$$

**THEOREM 6.5.** *Let  $\mathcal{A}$  be a simplicial hyperplane arrangement with base region  $C_0 \in \mathcal{R}(\mathcal{A})$  and  $h$ -polynomial  $h_{\mathcal{A}}(z)$ . Then*

$$f_{\mathcal{A}}(z) = h_{\mathcal{A}}(z + 1)$$

PROOF. It follows from Lemma 6.4 that for every face  $F$  of  $\mathcal{A}$ , there is a unique  $C \in \mathcal{R}(\mathcal{A})$  with  $F \subseteq \widehat{C}$ . Thus

$$f_{\mathcal{A}}(z) = \sum_{C \in \mathcal{R}(\mathcal{A})} f_{\widehat{C}}(z) = \sum_{C \in \mathcal{R}(\mathcal{A})} (1 + z)^{m - |D(C)|} = \sum_{k=0}^m h_k(\mathcal{A})(1 + z)^{m-k}. \quad \square$$

Our notation does not emphasize which base region we selected for the computation of the half-open decomposition or  $h_{\mathcal{A}}(z)$ . But it does not matter.

**Corollary 6.6.** *The  $h$ -polynomial is independent of the base region  $C_0$ .*

PROOF. Note that  $h_{\mathcal{A}}(z) = f_{\mathcal{A}}(z - 1)$ . □

**Corollary 6.7** (Dehn–Sommerville equations). *If  $\mathcal{A}$  is an essential simplicial arrangement in  $n$ -dimensional space, then  $h_{\mathcal{A}}(z) = z^n h_{\mathcal{A}}(\frac{1}{z})$ , that is,*

$$h_k(\mathcal{A}) = h_{n-k}(\mathcal{A})$$

PROOF. A wall of  $C$  separates  $C$  from  $C_0$  if and only if it does not separate  $C$  from  $-C_0$ . Thus, the  $h$ -polynomial with respect to  $-C_0$  as the base region is precisely

$$\sum_{C \in \mathcal{R}(\mathcal{A})} z^{|D(C)|} = z^n h_{\mathcal{A}}(\frac{1}{z}).$$

Since the  $h_{\mathcal{A}}(z)$  is independent of the choice of a base region now complete the proof. □

Note that we can express the  $h$ -polynomial of a reflection arrangement  $\mathcal{A}(W)$  algebraically as follows. Recall from (2) that  $\alpha \in \Delta$  is a **right descent** of  $w \in W$  if  $\ell(ws_{\alpha}) < \ell(w)$  and  $D_R(w)$  is the **descent set** of  $w$ . By the Exchange condition of Corollary 4.10,  $\alpha \in D_R(w)$  if and only if there is a reduced expression that ends in  $s_{\alpha}$ . Using Construction 4.5, this means that  $t := ws_{\alpha}w^{-1}$  is a reflection in a wall of  $wC_0$  that separates  $wC_0$  from  $C_0$ . That is

$$|D(wC_0)| = |D_R(w)|.$$

We write  $d(w) := |D_R(w)|$  and define the **Eulerian polynomial**

$$A_W(z) := \sum_{w \in W} z^{d(w)}.$$

The naming is inspired by the **Eulerian numbers**  $A(n, k)$  that count the number of permutations  $\sigma \in \mathfrak{S}_n$  with  $k$  descents, that is, indices  $1 \leq i < n$  with  $\sigma(i) > \sigma(i + 1)$ . For example

$$A_{\mathfrak{S}_3}(z) = 1 + 4z + z^2 \quad A_{\mathfrak{S}_4}(z) = 1 + 11z + 11z^2 + z^3;$$

see Figure 8.

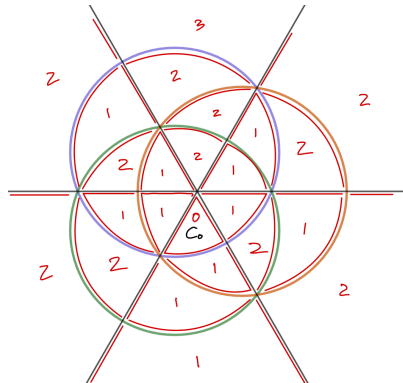


FIGURE 8. Half-open decomposition for reflection arrangement of symmetric group



### 7. Classification of finite reflection groups

In this section we will classify all the finite reflection groups. For that we will first seek a compact way to encode reflection groups. Let  $W$  be a reflection group that acts essentially on the  $n$ -dimensional real vector space  $V$ . Let  $\Phi$  be a root system consisting of vectors of unit length and let  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  be simple system. Remember that  $s_{\alpha_i}, s_{\alpha_j}$  for  $i \neq j$  generates a rank-2 reflection group  $\mathcal{D}_{m_{ij}}$  for some  $m_{ij} \geq 2$ . Since  $\alpha_i, \alpha_j$  is a simple system for  $\mathcal{D}_{m_{ij}}$ , we have that  $s_{\alpha_i} s_{\alpha_j}$  is a rotation by  $\frac{2\pi}{m_{ij}}$ . Hence  $(s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = e$  and

$$\langle \alpha_i, \alpha_j \rangle = -\cos \frac{\pi}{m_{ij}}.$$

We define the **Coxeter matrix**  $M = (m_{ij})_{i,j}$ . Let denote the Gram matrix of the simple system by  $A$  with entries

$$A_{ij} = \langle \alpha_i, \alpha_j \rangle = -\cos \frac{\pi}{m_{ij}}, \tag{4}$$

where  $m_{ii} = 1$  and  $A_{ii} = 1$ . Up to labelling the individual roots, the matrix  $A$  is independent of the choice of a simple system. It is more customary to record  $M$  by defining the **Coxeter graph**  $\Gamma$ , which is a simple undirected graph on nodes  $[n] = \{1, \dots, n\}$  such that  $ij$  is an edge iff  $m_{ij} \geq 2$ . The edge  $ij$  carries an edge label  $m_{ij}$  iff  $m_{ij} \geq 3$ .

We call  $W$  **irreducible** if the Coxeter graph is connected. For example the Coxeter graph of the dihedral group  $I_2(2)$  consists on two isolated nodes. If  $V_1 \uplus V_2 \uplus \dots \uplus V_k = [n]$  are the node sets of the connected components of  $\Gamma$ , then  $W_h = \langle s_i : i \in V_h \rangle$  defines a (standard) parabolic subgroup and  $W = W_1 \times W_2 \times \dots \times W_n$ . Hence, it suffices to classify the irreducible reflection groups. Figure 9 shows the Coxeter graphs for the dihedral groups as well as the groups of types  $A_n$  and  $B_n$  and some more.

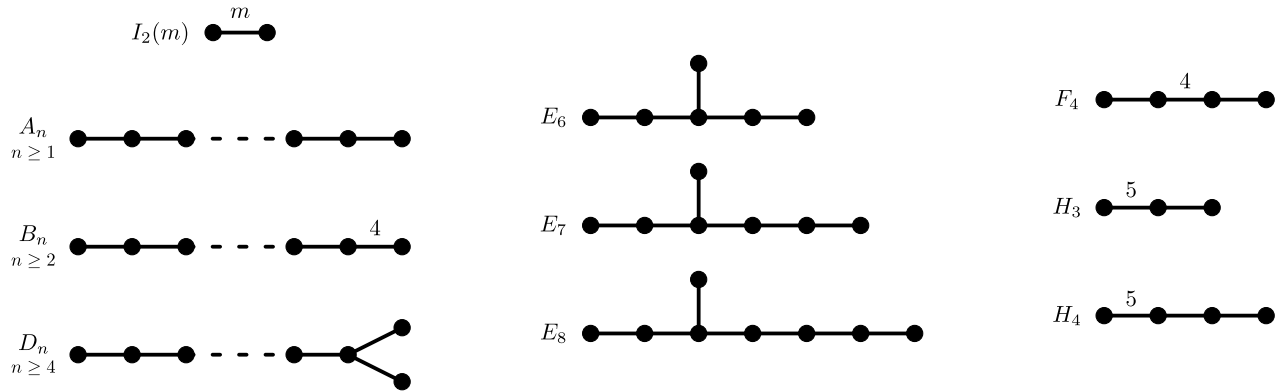


FIGURE 9. Coxeter graphs of all reflection groups.

The Coxeter graph is connected if and only if the Gram matrix  $A$  is **indecomposable**, that is, if there is no partition  $I \uplus J = [n]$  such that  $A_{ij} = 0$  for all  $i \in I, j \in J$ . Moreover, as a Gram matrix,  $A$  is symmetric and **positive semidefinite**, i.e.,  $x^t A x \geq 0$  for all  $x \in V$ . Moreover,  $A$  is **positive definite** if additionally  $x^t A x = 0$  implies  $x = 0$ .<sup>2</sup> We need the following lemma from matrix theory.

**Lemma 7.1.** *Let  $A$  be an indecomposable and positive semidefinite matrix such that  $A_{ij} \leq 0$  for all  $i \neq j$ .*

- a)  $\ker A = \{x : x^t A x = 0\}$
- b)  $\dim \ker A \leq 1$  and if  $\dim \ker A = 1$ , then  $\ker(A)$  is spanned by a vector with all entries positive.
- c) The eigenspace for the smallest eigenvalue is 1-dimensional and spanned by vector with strictly positive components.

<sup>2</sup>Yes, we should consistently write  $\langle x, Ax \rangle$  but this is too much.

PROOF. a) Since  $A$  is positive definite and hence the Gram matrix of some vector configuration, there is  $G$  with  $A = G^t G$  and hence  $x^t A x = \|Gx\|^2$ . Thus, if  $x^t A x = 0$ , then  $Gx = 0$  and hence  $Ax = 0$ . The direction  $Ax = 0$  implies  $x^t A x = 0$  is clear. (Here we used the positive semidefiniteness.)

b) Let  $x \in \ker A$  with  $x \neq 0$ . Define  $z$  by  $z_i = |x_i|$ . Then

$$0 \leq z^t A z = \sum_{i,j} |x_i| A_{ij} |x_j| \leq \sum_{i,j} x_i A_{ij} x_j = 0,$$

where we used  $A_{ij} \leq 0$ , we conclude that  $Az = 0$ . Define  $I := \{i : z_i = 0\}$  and  $J := [n] \setminus I$ . For every  $i$ , we have

$$0 = \sum_{j \in J} A_{ij} z_j.$$

Again, since  $A_{ij} \leq 0$  and  $z_j > 0$  for  $j \in J$ , we get  $A_{ij} = 0$  for all  $i \in I$  and  $j \in J$ . Since  $A$  is assumed to be irreducible, this means  $I = \emptyset$ . This shows that  $z > 0$ . In fact the argument shows that every  $x \neq 0$  with  $Ax = 0$  has to have all coordinates nonzero. However if there is  $x' \in \ker A$  linearly independent from  $x$ , then  $\alpha x + \beta x'$  has a zero component for some  $\alpha, \beta \in \mathbb{R}$ . This shows b).

For c), we note that if  $\lambda \geq 0$  is the smallest eigenvalue, then  $A' := A - \lambda I$  is still positive semidefinite, irreducible and non-positive off-diagonal entries. The kernel of  $A'$  is the eigenspace of  $A$  and is 1-dimensional by a).  $\square$

Every graph  $\Gamma$  on nodes  $[n]$  with edge labels  $m_{ij} \geq 3$  gives rise to a matrix  $A_\Gamma$  as defined (4). We call  $\Gamma$  positive (semi)definite if  $A_\Gamma$  is positive (semi)definite. A **subgraph**  $\Gamma'$  is obtained from  $\Gamma$  by deleting nodes together with incident edges or decreasing the label on some edges.

**Proposition 7.2.** *Let  $\Gamma$  be a connected, positive definite graph  $\Gamma$ , then every subgraph  $\Gamma'$  is positive definite.*

PROOF. Suppose first that  $\Gamma'$  is obtained by only deleting nodes and let  $J \subsetneq [n]$  be the node set of  $\Gamma'$ . The matrix  $A_{\Gamma'}$  is obtained from  $A_\Gamma$  by restricting to rows and columns indexed by  $J$ . In particular, if  $A_{\Gamma'}$  is not of full rank, then by Lemma 7.1a), there is an  $x$  with  $\text{supp}(x) = \{i : x_i \neq 0\} \subseteq J$  and  $x^t A_\Gamma x = 0$ . But if  $x \neq 0$ , then it follows from Lemma 7.1b) that  $J = [n]$ , which shows  $\Gamma = \Gamma'$  and  $\Gamma$  was positive definite.

Thus, it suffices to assume that  $\Gamma'$  has the same node set but  $m'_{ij} \leq m_{ij}$  for all edges  $ij$ . That is,  $A_{ij} = -\cos(\frac{\pi}{m_{ij}}) \leq -\cos(\frac{\pi}{m'_{ij}}) = A'_{ij}$ . Now if there is  $x \neq 0$  with  $x^t A' x \leq 0$ , then we compute

$$0 \leq \sum_{i,j} a_{ij} |x_i| |x_j| \leq \sum_{i,j} a'_{ij} |x_i| |x_j| \leq \sum_{i,j} a'_{ij} x_i x_j \leq 0.$$

The vector  $z$  of absolute values of  $x$  thus satisfies  $z^t A z = 0$ , which contradicts our assumption that  $\Gamma$  and hence  $A = A_\Gamma$  is positive definite.  $\square$

**THEOREM 7.3.** *The only connected, positive definite graphs are those in Figure 9.*

**PROOF. 1.  $\Gamma$  is a tree (i.e. does not contain a cycle).**

The matrix  $A$  of a cycle with all edge labels  $m_{ij} = 3$  consists of cyclic shifts of the row  $(\rho, 1, \rho, 0, \dots, 0)$ , where  $\rho = -\cos(\frac{\pi}{3}) = -\frac{1}{2}$ . Hence  $\mathbf{1}^t A \mathbf{1} = 0$  and a cycle cannot occur as a subgraph.

This means that  $\Gamma$  is a tree.

If  $\Gamma$  is not of type  $A_n$ , then it has node of degree  $\geq 3$ .

**2.  $\Gamma$  has no node of degree  $\geq 4$ .**

Otherwise, we could find as a subgraph, the graph  $\Gamma'$  with nodes  $1, 2, 3, 4, 5$  and edges  $i5$  for  $i = 1, \dots, 4$ .

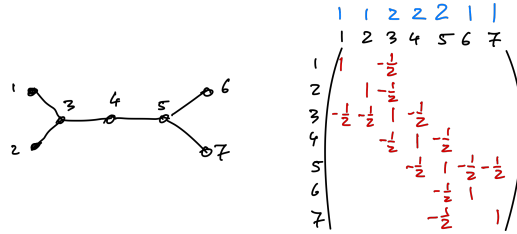
The corresponding matrix  $A_{\Gamma'}$  is

$$\begin{pmatrix} 1 & & & & -\frac{1}{2} \\ & 1 & & & -\frac{1}{2} \\ & & 1 & & -\frac{1}{2} \\ & & & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

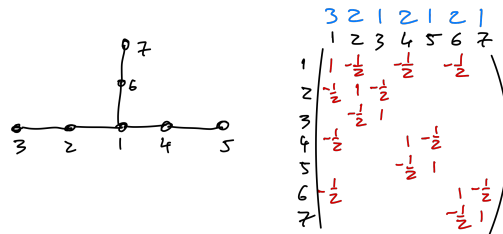
which is not positive definite.

**2.  $\Gamma$  has a unique node of degree = 3**

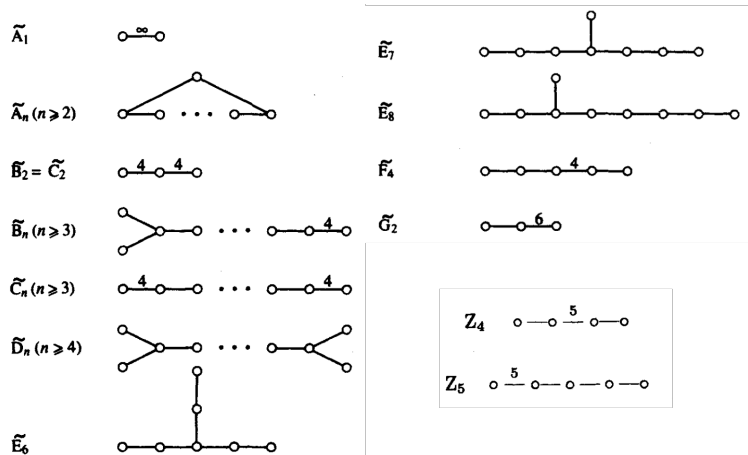
The following illustrates the situation and hints at a generalization



**3. The three paths starting at the branch point cannot be all  $\geq 2$  long.** The following illustrates the situation



In a similar manner, one checks that the following subgraphs cannot occur<sup>3</sup>:



□

This shows that if  $W$  is a finite reflection group, then its Coxeter graph has to belong to one of the 10 types shown in Figure 9. To construct a finite reflection group for each of the 10 types, let  $\Gamma$  be one of these on  $n$  nodes. The matrix  $A = A_{\Gamma} \in \mathbb{R}^{n \times n}$  is positive definite and there is a factorization  $A = G^t G$  for some full-rank matrix  $G \in \mathbb{R}^{n \times n}$ . Let  $\alpha_1, \dots, \alpha_n$  be the column vectors. Note that  $A_{ii} = 1$  and all

<sup>3</sup>The figure is taken from [3]

$\alpha_i$  are unit vectors. Define hyperplanes  $H_i = H_{\alpha_i}$  and reflections  $s_i = s_{\alpha_i}$  and let  $W$  be the subgroup of  $\text{GL}(\mathbb{R}^n)$  generated by these reflections. Using the fact that  $\langle \alpha_i, \alpha_j \rangle \leq 0$  for all  $i \neq j$  we can employ ideas similar to those of Section 3 to show that for every  $\alpha_i$ , the orbit  $W\alpha_i$  is finite. If we let  $\Phi$  be the union of all these finite orbits, then this allows us to show that  $W \rightarrow \mathfrak{S}(\Phi)$  is a monomorphism and hence  $W$  is finite; cf. Exercise 2.3.

**THEOREM 7.4.** *The irreducible finite reflection groups are, up to isomorphism, in one-to-one correspondence with the Coxeter graphs of Figure 9.*

## Arrangements and simpliciality

The beginning of this chapter essentially follows the first chapters of the book *Arrangements of Hyperplanes* by Orlik and Terao [4].

### 1. Arrangements of hyperplanes

**Definition 1.1.** Let  $K$  be a field,  $\ell \in \mathbb{N}$ , and  $V := K^\ell$ . An **arrangement of hyperplanes** (or  **$\ell$ -arrangement**)  $(\mathcal{A}, V)$  (or  $\mathcal{A}$  for short) is a finite set of hyperplanes  $\mathcal{A}$  in  $V$ .

**Example 1.2.** Let  $K = \mathbb{F}_q$  be a finite field and  $V = \mathbb{F}_q^\ell$ . The set of all linear hyperplanes in  $V$  is an arrangement of hyperplanes.

**Remark 1.3.** Let  $e_1, \dots, e_\ell$  be a basis of  $V$  and  $x_1, \dots, x_\ell \in V^*$  be the dual basis. The symmetric algebra  $S(V^*)$  is a polynomial algebra  $K[x_1, \dots, x_\ell]$ .

Each hyperplane  $H$  in an arrangement is the kernel of a linear form  $\alpha_H$  (= polynomial of degree 1 in  $S(V^*)$ ). However, for any unit  $a \in K^\times$ , the linear form  $a \cdot \alpha_H$  defines the same hyperplane. So in a certain sense, an arrangement of hyperplanes may equivalently be defined as a set of points in the projective space  $\mathbb{P}(V^*)$ .

Since  $\alpha_H$  is defined only up to scalars, it will be convenient to write

$$f \doteq g \quad \Leftrightarrow \quad \exists a \in K^\times : f = a \cdot g.$$

**Definition 1.4.** Let  $\mathcal{A}$  be an arrangement and write  $\alpha_H, H \in \mathcal{A}$  for defining linear forms. The product

$$Q(\mathcal{A}) \doteq \prod_{H \in \mathcal{A}} \alpha_H$$

is called the **defining polynomial** of  $\mathcal{A}$ .

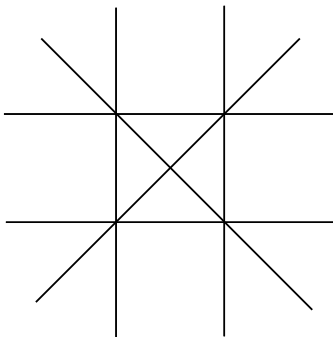


FIGURE 1. The arrangement of type  $A_3$ .

**Example 1.5.** The arrangement  $\mathcal{A}$  of type  $A_3$  has 6 hyperplanes in  $V = \mathbb{R}^3$  (see Figure 1). With respect to the standard basis  $x, y, z$  of  $V^*$ , the hyperplanes of  $\mathcal{A}$  are the kernels of the linear forms

$$(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 1, 1).$$

We have  $S(V^*) \cong K[x, y, z]$  and

$$Q(\mathcal{A}) = xyz(x+y)(y+z)(x+y+z).$$

**Example 1.6.** The arrangement in  $V = K^\ell$  defined by

$$Q(\mathcal{A}) = \prod_{i=1}^{\ell} x_i$$

is called the **boolean arrangement**.

**Example 1.7.** For  $1 \leq i < j \leq \ell$  let  $H_{i,j} = \ker(x_i - x_j)$ . The arrangement in  $V = \mathbb{R}^\ell$  defined by

$$Q(\mathcal{A}) = \prod_{1 \leq i < j \leq \ell} (x_i - x_j)$$

consisting of all the hyperplanes  $H_{i,j}$  is called the **braid arrangement**.

**Exercise 1.8.** Compute the number  $|\mathcal{A}|$  of hyperplanes for the arrangements in examples 1.6, 1.7, 1.2.

**Definition 1.9.** Let  $\mathcal{A}$  be an arrangement. The set

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in U} H \mid U \subseteq \mathcal{A} \right\}$$

is called the **intersection lattice** of  $\mathcal{A}$ . It is partially ordered by reverse inclusion:

$$X \leq Y \iff Y \subseteq X, \quad \text{for } X, Y \in L(\mathcal{A}).$$

If  $X \in L(\mathcal{A})$ , then the **rank**  $r(X)$  of  $X$  is defined as  $r(X) := \ell - \dim X$ , i.e. the codimension of  $X$  and the rank of the arrangement  $\mathcal{A}$  is defined as  $r(\mathcal{A}) := r(T(\mathcal{A}))$  where  $T(\mathcal{A}) := \bigcap_{H \in \mathcal{A}} H$  is the **center** of  $\mathcal{A}$ . The arrangement  $\mathcal{A}$  is called **central** if  $0 \in H$  for all  $H \in \mathcal{A}$ . An  $\ell$ -arrangement  $\mathcal{A}$  is called **essential** if  $r(\mathcal{A}) = \ell$ .

**Example 1.10.** The braid arrangement  $\mathcal{A}$  is not essential:

$$T(\mathcal{A}) = \langle (1, \dots, 1) \rangle, \quad r(\mathcal{A}) = \ell - \dim(T(\mathcal{A})) = \ell - 1.$$

**Example 1.11.** Recall the arrangement  $\mathcal{A}$  of type  $A_3$ . It is essential and of rank 3. The intersection lattice consists of  $0, V, H \in \mathcal{A}$  and the subspaces generated by

$$(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, -1), (1, 0, -1), (1, -1, 0), (1, -1, 1).$$

**Definition 1.12.** Let  $\mathcal{A}$  be an arrangement. For  $X \in L(\mathcal{A})$ , we define the **localization**

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$$

of  $\mathcal{A}$  at  $X$ , and the **restriction of  $\mathcal{A}$  to  $X$** ,  $(\mathcal{A}^X, X)$ , where

$$\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

**Example 1.13.** Localizations and restrictions in the arrangement of type  $A_3$  (Picture).

**Definition 1.14** ([4, 2.13, 2.15]). Let  $(\mathcal{A}_1, V_1)$  and  $(\mathcal{A}_2, V_2)$  be arrangements. The **product** of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  is the arrangement  $(\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$  where

$$\mathcal{A}_1 \times \mathcal{A}_2 := \{H \oplus V_2 \mid H \in \mathcal{A}_1\} \cup \{V_1 \oplus H \mid H \in \mathcal{A}_2\}.$$

An arrangement  $(\mathcal{A}, V)$  is called **reducible** if there exist arrangements  $(\mathcal{A}_1, V_1)$  and  $(\mathcal{A}_2, V_2)$  such that  $(\mathcal{A}, V) = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$ . Otherwise  $(\mathcal{A}, V)$  is called **irreducible**.

**Example 1.15.** The arrangement in Figure 2 is called a **near pencil**. It is reducible.

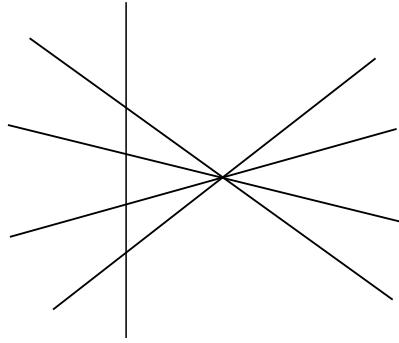


FIGURE 2. A near pencil arrangement.

### 2. Simplicial arrangements

**Definition 2.1.** Let  $\ell \in \mathbb{N}$ ,  $V := \mathbb{R}^\ell$ , and  $\mathcal{A}$  an arrangement in  $V$ . Let  $\mathcal{K}(\mathcal{A})$  be the set of connected components (**chambers**) of  $V \setminus \bigcup_{H \in \mathcal{A}} H$ . If every chamber  $K$  is an **open simplicial cone**, i.e. there exist  $\alpha_1^\vee, \dots, \alpha_\ell^\vee \in V$  such that

$$K = \left\{ \sum_{i=1}^{\ell} a_i \alpha_i^\vee \mid a_i > 0 \text{ for all } i = 1, \dots, \ell \right\} =: \langle \alpha_1^\vee, \dots, \alpha_\ell^\vee \rangle_{>0},$$

then  $\mathcal{A}$  is called a **simplicial arrangement**.

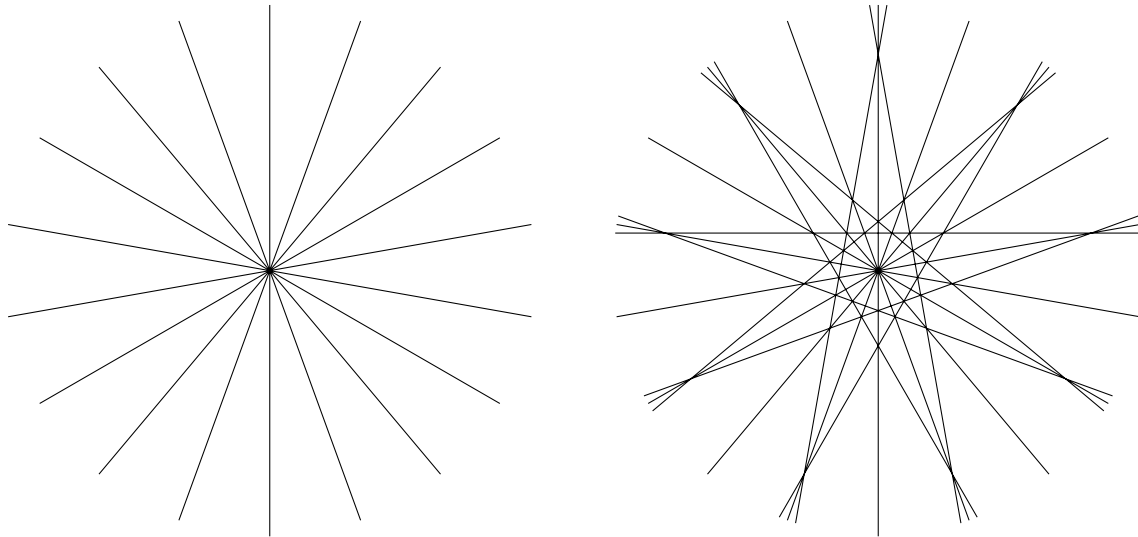


FIGURE 3. A simplicial arrangement in  $\mathbb{R}^2$ , a representation of a simplicial arrangement in  $\mathbb{R}^3$  in the projective plane.

**Example 2.2.** (1) Figure 3 displays examples for  $\ell = 2$  and  $\ell = 3$ .

(2) Let  $W$  be a real reflection group,  $R$  the set of roots of  $W$ . For  $\alpha \in V^*$  we write  $\alpha^\perp = \ker(\alpha)$ . Then  $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$  is a simplicial arrangement.

**Theorem 2.3** (Deligne, 1972). The complement of a complexified finite simplicial arrangement is  $K(\pi, 1)$ .

For dimension three, there is a catalogue of known simplicial arrangements [2] and we have a complete list of simplicial arrangements with at most 27 lines [1]. There are a little less than 100 “sporadic” arrangements in the catalogue. For most of them, we have no satisfactory explanation yet. H.S.M. Coxeter writes

“[...] the diagrams which profess to portray these known polygrams are strangely unintelligible.”





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# Index

- $\ell$ -arrangement, 29
- $f$ -polynomial, 21
- $f$ -vector, 21
- $h$ -polynomial, 23
- $q$ -integer, 17
- (right) descent, 19
  
- acyclic, 9
- adjacent, 5, 13
- arrangement of hyperplanes, 29
  
- boolean arrangement, 30
- braid arrangement, 30
  
- center, 30
- central, 30
- chambers, 12, 31
- codimension, 20
- conical hull, 9
- Coxeter graph, 25
- Coxeter matrix, 25
- cycle notation, 5
  
- defining polynomial, 29
- descent set, 24
- dihedral group, 7
  
- essential, 6, 30
- Eulerian numbers, 24
- Eulerian polynomial, 24
  
- face, 20
- face vector, 21
- facet, 13
- finite reflection group, 6
- fixed space, 6
- flag, 8
- fundamental domain, 16
  
- gallery, 13
- generic, 9
  
- half-open, 22
- half-open cone, 23
- height, 11
  
- indecomposable, 25
- intersection lattice, 30
- inversion, 5
- inversion number, 5
- irreducible, 9, 25, 30
  
- isotropy group, 17
  
- length, 5, 15
- length generating polynomial, 17
- linear hyperplane, 6
- localization, 30
- longest element, 16, 20
  
- minimal coset representatives, 18
  
- near pencil, 30
  
- one-line, 5
- open, 12
- open simplicial cone, 31
  
- panel, 13
- parabolic subgroup, 17
- polyhedral cone, 12
- positive halfspace, 12
- positive semidefinite, 25
- positive system, 9
- positive definite, 25
- product, 30
  
- rank, 6, 8, 30
- reduced expression, 5, 15
- reducible, 9, 30
- reflection, 6
- regions, 12
- regular, 8
- restriction of, 30
- right descent, 24
- root system, 8
- roots, 8
  
- second kind, 22
- separates, 13
- simple, 5
- simple system, 10
- simplicial, 12
- simplicial arrangement, 31
- simply transitive, 13
- stabilizer group, 17
- standard parabolic subgroup, 17
- Stirling numbers, 22
- subgraph, 26
- symmetric group, 5
  
- transposition, 5
- transpositions, 5

type, 8

valid, 20

wall, 13