WiSe 2024/25
Prof. Dr. Lukas Kühne

UNIVERSITÄT BIELEFELD
Faculty of Mathematics

Algebraic and Topological Methods in Discrete Mathematics

Finite reflection groups, hyperplane arrangements, and (oriented) matroids

13. Homework sheet

Problem 1.
(a) Which of the 15 lattices on 6 elements are graded?
(b) Which of the graded lattices on 6 elements admit an R-labeling?

Problem 2.
(a) Fix any order and determine the broken circuit complex of the uniform matroid $U_{m,n}$ for $1 \leq m \leq n$.
(b) Determine the characteristic polynomial $\chi_{U_{m,n}}(t)$ of the uniform matroid $U_{m,n}$.

Problem 3. Consider again the lattice Π_n of all partitions of the set $\{1, \ldots, n\}$. Note that the cover relations in lattice are of the form:

$$\{B_1, B_2, B_3 \ldots, B_k\} <: \{B_1 \cup B_2, B_3, \ldots, B_k\},$$

that is they exactly arise by merging two blocks in a partition.

Let $\lambda : \mathrm{Cov}(\Pi_n) \to \mathbb{Z}$ be defined by setting $\lambda(\pi_1, \pi_2) := \max(\min(B_1), \min(B_2))$ for two partitions $\pi_1 <: \pi_2$ which differ by merging the two blocks B_1 and B_2. Prove that this labeling is an R-labeling on the partition lattice Π_n.

Problem 4. For this exercise I recommend to use the matroid package in `sagemath`. It has a built-in interface to the database of matroids as described here.
(a) How many simple matroids are there on the ground set $\{1, \ldots, 9\}$ of rank 3?
(b) What is the minimum and maximum of the Whitney number w_2 amongst these matroids?
(c) Which of these matroids satisfy the combinatorial criterion of being simplicial as in Definition 3.14 in the lecture notes?
(d) Does the characteristic polynomial have only real roots for all matroids?