Algebraic and Topological Methods in Discrete Mathematics

Finite reflection groups, hyperplane arrangements, and (oriented) matroids

10. Homework sheet

Problem 1. For $c \in \mathbb{Z}$, let

$$\eta(c) := \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}.$$

Notice that up to a transposition, $\eta(c)$ may be viewed as a reflection:

$$\eta(c)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & c \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\eta(c) = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}.$$

Hence a composition of reflections in a Weyl groupoid of rank two corresponds to a product of matrices of the form $\eta(c)$; in particular, starting with a chamber and reflecting to the opposite chamber corresponds to a product

$$\prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $c_1, \ldots, c_m \in \mathbb{Z}_{\geq 0}$.

A sequence $(c_1, \ldots, c_m)$ with $\prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is called a *quiddity cycle*.

Prove the following results on quiddity cycles:

(a) If $(c_1, \ldots, c_m)$ is a quiddity cycle, then $(c_m, \ldots, c_1)$ and $(c_2, \ldots, c_m, c_1)$ are quiddity cycles.

(b) The cycles $(0, 0)$ and $(1, 1, 1)$ are the only quiddity cycles of length 2 and 3.

(c) If $(c_1, \ldots, c_m) \in \mathbb{Z}_{>0}^m$ is a quiddity cycle, then there exists an i with $c_i = 1$.

(d) For any a, b we have $\eta(a)\eta(b) = \eta(a+1)\eta(1)\eta(b+1)$.

(e) We now consider only quiddity cycles $(c_1, \ldots, c_m)$ with the following additional property: Let

$$M_{i,j} = \prod_{k=i}^{j} \eta(c_k).$$

Then we assume that $(M_{i,j})_{1,1} \neq 0$ for all $i < j + 2 < i + m$, where the index k of c_k is read modulo m.

Conclude that such quiddity cycles with entries in $\mathbb{Z}_{>0}$ correspond to triangulations of convex polygons by non-intersecting diagonals.

We obtain that crystallographic arrangements of rank two are parametrized by triangulations of convex polygons by non-intersecting diagonals.

(f) Explain how the Weyl groups of types $A_2, B_2/C_2, G_2$ correspond to the cycles $(1, 1, 1), (1, 2, 1, 2), (1, 3, 1, 3, 1, 3)$.