Algebraic and Topological Methods in Discrete Mathematics

Finite reflection groups, hyperplane arrangements, and (oriented) matroids

9. Homework sheet

Problem 1. Let $\mathcal{A}$ be be a simplicial arrangement in $\mathbb{R}^3$.

(a) If there are $X, Y \in L(\mathcal{A})$ with $r(X) = r(Y) = 2$, $|\mathcal{A}_X| = |\mathcal{A}_Y| = 2$ and such that $X \neq Y$ are adjacent to the same chamber, then $\mathcal{A}$ is reducible.

(b) If $\mathcal{A}$ is irreducible, then there are $X, Y \in L(\mathcal{A})$ with $r(X) = r(Y) = 2$ such that X, Y are adjacent to the same chamber and $|\mathcal{A}_X| = 2$, $|\mathcal{A}_Y| = 3$.

Problem 2. Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_q^3$. Then $\mathcal{A}$ is simplicial if and only if

$$|\mathcal{A}| = 3q - \frac{3n_0^{\mathcal{A}} + 2n_1^{\mathcal{A}}}{q+1}$$

where

$$n_i^{\mathcal{A}} := |\{\langle v\rangle \mid v \in \mathbb{F}_q^3 \setminus \{0\},\ \ |\{H \in \mathcal{A} \mid v \in H\}| = i\}|.$$

In particular, if $\mathcal{A}$ is simplicial, then $|Ac| \leq 3q$.

Problem 3. Let $\mathbb{F}_q$ be a finite field,

$$\begin{aligned} \mathcal{A} &:= \{(0,1,a)^\perp \mid a \in \mathbb{F}_q\} \cup \{(1,a,a^2)^\perp \mid a \in \mathbb{F}_q\}, \quad \text{and} \\ \mathcal{E} &:= \{(1,a,0)^\perp \mid a \in \mathbb{F}_q^\times\}, \end{aligned}$$

where $(a,b,c)^\perp$ denotes the kernel of the matrix $(a\,b\,c)$.

Show that if q is odd, then $\mathcal{A} \cup \mathcal{B}$ is simplicial for all $\mathcal{B} \subseteq \mathcal{E}$.

Problem 4. Is this arrangement simplicial?