GOETHE UNIVERSITÄT FRANKFURT AM MAIN

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Algebraic and Topological Methods in Discrete Mathematics

Finite reflection groups, hyperplane arrangements, and (oriented) matroids

6. Homework sheet

Problem 1. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ and $\mathcal{F} = \mathcal{F}(\mathcal{A})$ the collection of faces of $\mathcal{A}$, that is, the set of faces of the regions of $\mathcal{A}$. The **Euler characteristic** of $\mathcal{F}$ is

$$\chi(\mathcal{F}) \ := \ \sum_{F \in \mathcal{F}} (-1)^{\dim F} .$$

How that if $\mathcal{A}$ is simplicial[1], then $\chi(\mathcal{F}) = (-1)^n$.

Problem 2. For a permutation $\sigma \in \mathfrak{S}_n$, a **descent** is an index $1 \le i < n$ with $\sigma(i) > \sigma(i+1)$. We write $\mathrm{Des}(\sigma) \subseteq [n-1]$ for the descents of σ. For $S \subseteq [n-1]$, define $\mathrm{Des}_S := \{\sigma \in \mathfrak{S}_n : \mathrm{Des}(\sigma) = S\}$ and define $\mathrm{Des}^S := \{\sigma \in \mathfrak{S}_n : \mathrm{Des}(\sigma) \subseteq S\}$.

(a) For $S = \{s_1 < s_2 < \cdots < s_k\}$ show that

$$|\mathrm{Des}^S| \ = \ \binom{n}{s_1, s_2 - s_1, \dots, s_k - s_{k-1}, n - s_k} \ = \ \frac{n!}{s_1!(s_2-s_1)!\cdots(s_k-s_{k-1})!(n-s_k)!} .$$

(b) Define the multivariable polynomials $F, f \in \mathbb{R}[x_1, \dots, x_{n-1}]$ by

$$f(x_1,\dots,x_{n-1}) \ := \ \sum_{S \subseteq [n-1]} \mathrm{Des}_S(S) \prod_{i \notin S} x_i$$

$$F(x_1,\dots,x_{n-1}) \ := \ \sum_{S \subseteq [n-1]} \mathrm{Des}^S(S) \prod_{i \notin S} x_i .$$

Show that $F(x_1,\dots,x_{n-1}) = f(x_1+1,\dots,x_{n-1}+1)$.

Problem 3. Let W be an essential[2] finite reflection group acting on V and let $T = \{s_\beta : \beta \in \Phi\}$ be the collection of *all* reflections in W. Define the **absolute length** of $w \in W$ as

$$\overline{\ell}(w) \ := \ \min\{k : w = t_1 t_2 \cdots t_k : t_1, \dots, t_k \in T\} .$$

(a) For $w \in W$, let $V^w := \{v \in V : wv = v\}$ be the subspace fixed by w. Show that $\overline{\ell}(w) = \dim V - \dim V^w$.

(b) This implies that $\overline{\ell}(w) \le \dim V =: n$. Show that if Δ is a simple system, then the product of $s_\alpha, \alpha \in \Delta$ in any order achieves the maximal absolute length.

(c) For $W = \mathfrak{S}_n$, show that $\overline{\ell}(\sigma) = n - k$ iff σ is the product of k disjoint cycles. Note that cycles of length 1 (i.e. fixed points) are also counted. For example $e = (1)(2)(3) \in \mathfrak{S}_3$ has absolute length $\overline{\ell}(e) = 0 = 3 - 3$.

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[1] It also holds for general arrangements but it's particularly nice and easy for simplicial ones.

[2] Fixed space of W is 0.