GOETHE UNIVERSITÄT FRANKFURT AM MAIN

WiSe 2024/25
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Algebraic and Topological Methods in Discrete Mathematics

Finite reflection groups, hyperplane arrangements, and (oriented) matroids

5. Homework sheet

Problem 1. Find an example of a reflection group W and a subgroup $W' \subset W$ that is generated by reflections but that is not a parabolic subgroup.

Problem 2. For a permutation $\tau \in \mathfrak{S}_n$ define $I(\tau) = (a_1, \dots, a_n)$ by $a_i := \#\{j > i : \tau(i) > \tau(j)\}$. This defines a map $I : \mathfrak{S}_n \to [n-1] \times [n-2] \times \cdots \times [2] \times [1]$.

(a) Show that I is injective. Hint: If $I(\sigma) = I(\tau)$, consider the maximal t with $\sigma^{-1}(t) \neq \tau^{-1}(t)$.

(b) Conclude that I is bijective.

(c) Prove that

$$\mathfrak{S}_n(q) \;=\; [n]_q[n-1]_q \cdots [2]_q[1]_q \;=:\; [n]_q!\,.$$

Problem 3. For a finite reflection group W, let $T \subseteq W$ the set of all reflections. Determine

$$T(q) \;=\; \sum_{t \in T} q^{\ell(t)}\,.$$

for the symmetric group $W = \mathfrak{S}_n$. What can you say for general reflection groups W?