GOETHE UNIVERSITÄT FRANKFURT AM MAIN

WiSe 2024/25
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Algebraic and Topological Methods in Discrete Mathematics

Finite reflection groups, hyperplane arrangements, and (oriented) matroids

4. Homework sheet

Problem 1. Prove Corollary 5.10.

Problem 2. Let W be a finite reflection group with simple system Δ and length function ℓ relative to Δ.

(a) Show that there is a unique element $w_0 \in W$ of maximal length. This is called the **longest element** of W relative to Δ. What is the length?

(b) Show that w_0 is an involution, that is $w_0^2 = e$.

(c) Prove that in every reduced expression of w_0, every simple reflection must occur at least once.

(d) Let w with reduced expression $w = s_1 \cdots s_r$. Show that there is a w' with reduced expression $w' = s_{r+1} \cdots s_m$ such that $s_1 \cdots s_r s_{r+1} \cdots s_m$ is a reduced expression for w_0.

(e) ~~Show that for every $w \in W$ there is a simple system Δ such that w is the longest element.~~

Problem 3. For a collection of linear hyperplanes $\mathcal{A}$ in V, the **lattice of flats** $\mathcal{L} = \mathcal{L}(\mathcal{A})$ is the collection of linear subspaces obtained as intersections of hyperplanes in $\mathcal{A}$, partially ordered by *reverse* inclusion. Thus, the minimal element in $\mathcal{L}$ is V, the maximal element is $\bigcap \mathcal{A}$.

Let $\mathcal{A}$ be the reflection arrangement for the reflection group of type A_{n-1}.

(a) Give a *combinatorial*[1] description of the elements in $\mathcal{L}$.

(b) How many elements $L \in \mathcal{L}$ of codimension k are there? What famous combinatorial numbers count the number of codim k subspaces?

(c) Show that all parabolic subgroups are products of symmetric groups.

[1] Here, this should mean that you can give a description in the language of finite sets and no mention of (linear) algebra or geometry. You'll see!