GOETHE UNIVERSITÄT FRANKFURT AM MAIN

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Algebraic and Topological Methods in Discrete Mathematics

Finite reflection groups, hyperplane arrangements, and (oriented) matroids

3. Homework sheet

Problem 1. For $u \in V$, we define the segment $[-u, u] := \{\mu u : |\mu| \leq 1\}$. For $U = \{u_1, \ldots, u_m\} \subset V$, the **zonotope** is the compact convex set

$$Z = Z(U) \ := \ \sum_{i=1}^{m}[-u_i, u_i] \ = \ \{\mu_1 u_1 + \cdots + \mu_m u_m : |\mu_i| \leq 1 \text{ for all } i = 1, \ldots, m\}\,.$$

A point $p \in Z$ is a **vertex** if there is some $c \in V$ such that $\langle c, p\rangle > \langle c, q\rangle$ for all $q \in Z \setminus p$.

(a) Let Φ be a root system. Show that the vertices of $Z(\Phi)$ are in bijection to positive systems.

(b) Consider the standard root systems Φ of types A_{n-1} and B_n and determine the vertices of $Z(\Phi)$.

(c) $Z(U)$ is **inscribed** if there is a constant $\kappa > 0$ such that $\kappa = \langle v, v\rangle$ for all vertices $v \in Z(U)$. Show that $Z(\Phi)$ is alsways inscribed.

Problem 2. Let $\mathcal{A}$ be a collection of finitely many hyperplanes in V and $\mathcal{R} = \mathcal{R}(\mathcal{A})$ its set of regions. For any two regions $C, C' \in \mathcal{R}$ write $n(C.C')$ for the number of hyperplanes separating C from C'.

(a) Show that n defines a metric on $\mathcal{R}$.

(b) Let $p \in C^\circ$ and $q \in (C')^\circ$ be generic show that $n(C, C')$ is the number of hyperplane $H \in \mathcal{A}$ that meet $[p, q]$.

(c) Let $C = C_0, C_1, \cdots, C_k = C'$ be a gallery and $H_i = \operatorname{span}(C_{i-1} \cap C_i)$ the walls along the gallery. Show that if $k > n(C, C')$, then there are $i < j$ such that $H_i = H_j$.

Problem 3. Let $\mathcal{A}(W)$ be the reflection arrangement of a finite reflection group W. Define a graph G on nodes $\mathcal{R}(W)$ with $CC' \in E(G)$ if C, C' are adjacent.

(a) Show that G is **bipartite**, that is, there is partition $\mathcal{R}(W) = A \uplus B$ such A and B do not contain adjacent chambers.

(b) Prove that $|A| = |B|$.

Problem 4. Let G be a general group. For $x, y \in G$ write $x^y := yxy^{-1}$. Let $r_1, \ldots, r_l \in G$ be involutions. Prove that

$$r_1 r_2 \cdots r_l \ = \ r_l^{r_1 \cdots r_{l-1}} r_{l-1}^{r_1 \cdots r_{l-2}} \cdots r_2^{r_1} r_1\,.$$