GOETHE UNIVERSITÄT FRANKFURT AM MAIN

WiSe 2024/25
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Algebraic and Topological Methods in Discrete Mathematics

Finite reflection groups, hyperplane arrangements, and (oriented) matroids

2. Homework sheet

Problem 1. Let Φ be an irreducible root system. Show that there are at most two different lengths of roots. (Hint: Show this first for root systems of rank 2.)

Problem 2. Let $\Phi = \{\pm(e_j - e_i) : 1 \leq i < j \leq n\}$ be the root system of type A_{n-1}. Every subsets $U \subseteq \Phi$ determines a directed graph $D = (V, A)$ on nodes $V = \{1, \dots, n\}$ and arcs $A = \{(i, j) : e_i - e_j \in U\}$.

(a) Show that U generates $\mathfrak{S}_n$ if and only if D is connected.
(b) Show that any minimal generating set U has the same cardinality.
(c) How many minimal generating sets U are there?
(d) How many simple systems are contained in Φ?

Problem 3. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a simple system of a finite reflection group W. Let $\beta_1, \dots, \beta_n \in V$ be vectors with $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle$ for all $i, j = 1, \dots, n$. Show that s_{β_i} generates a finite reflection group that is conjugate to W, that is, equal to gWg^{-1} for some $g \in O(V)$.

Problem 4. Let Φ be a root system with simple system $\Delta = \{\alpha_1, \dots, \alpha_n\}$. For every $\beta \in \Phi^+$, let $\beta = a_1\alpha_1 + \cdots + a_n\alpha_n$ be the unique representation with $a_1, \dots, a_n \geq 0$. Define the height as $ht(\beta) = a_1 + \cdots + a_n$. Show that $ht(\beta) > 1$ if and only if $\beta \in \Phi^+ \setminus \Delta$.